RESEARCH STATEMENT

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I am a geometer, and I am interested in the geometric topology of manifolds and in geometric group theory. Manifolds and their symmetries are ubiquitous throughout in mathematics and its applications. Surfaces were studied already in the nineteenth century by Riemann, and higher dimensional manifolds became central of objects in the twentieth beginning with the work of Poincaré, and continuing with the work of Thom, Milnor, Wall, Smale, Novikov—to name just a few.

My work makes use of the techniques and ideas from algebraic and combinatorial topology, surgery theory, number theory, and complexity theory. A major theme of much of my work is the extent to which an object with some singularities or deficiencies can be "resolved" or "improved" to a nicer object. For instance,

- in Section 1.1, there is the question of whether the classifying space for a $\mathbb{Q}$-PD group can be "resolved" to an aspherical closed $\mathbb{Q}$-homology manifold;
- in Section 1.2, there is the question of whether an aspherical topological manifold can be "resolved" by a simplicial complex;
- in Section 2.1, there is the question of whether a PL $\mathbb{Q}$-homology manifold M having cohomology ring $H^\star(M;\mathbb{Q}) = \mathbb{Q}[x]/(x^3)$ can be resolved by a smooth manifold; and
- in Section 2.4, there is the issue of whether, say, a cochain can be replaced by a "polynomially bounded" cochain.

To make progress on questions like these, I make significant use of computational techniques, and often use the computer algebra system `sage` to run experiments. My research has three specific directions:

- aspherical manifolds and Poincaré duality groups,
- computations surrounding high-dimensional manifolds, and
- undergraduate research.

By "undergraduate research," I mean research projects involving undergraduates (some of which are listed on Page 8) but also research projects which occur as part of online courses and my NSF TUES grant. As of November 2013, there have been 147k enrollments in my online courses, which have resulted in significant amounts of data. This project is discussed in Section 3.4.

1. Aspherical manifolds and Poincaré duality groups

1.1. **Rational Poincaré duality.** A group G is a Poincaré duality group if its classifying space BG satisfies Poincaré duality; examples include fundamental groups of aspherical manifolds. C. T. C. Wall asked whether a Poincaré duality group is necessarily the fundamental group of an aspherical manifold, and M. Davis in [**MR1747535**] broadened Wall's question to R-homology manifolds:

Question 1. Is every finitely presented torsion-free group satisfying Poincaré duality with R-coefficients ("R-PD") the fundamental group of an aspherical closed R-homology manifold?

My thesis answers M. Davis' question in the negative, using Bestvina–Brady Morse theory to produce $\mathbb{Q}$-PD groups G for which BG does not have the homotopy type of a finite complex.

I also addressed a generalization version of M. Davis' question. There are weaker geometric conditions that nonetheless imply a group is R-PD: for π to satisfy Poincaré duality with R coefficients, it suffices that π act freely on an R-acyclic R-homology manifold.

Question 2. Does every finitely presented group π satisfying Poincaré duality with R coefficients act freely on an R-acyclic R-homology manifold?

This is interesting when $R = \mathbb{Q}$. All finite groups are 0-dimensional $\mathbb{Q}$-PD, and extensions of $\mathbb{Q}$-PD groups by finite groups are therefore $\mathbb{Q}$-PD. The restriction to torsion-free groups is not necessary, as groups with torsion may act freely on $\mathbb{Q}$-acyclic (albeit not contractible) finite complexes.

A particular class of $\mathbb{Q}$-PD groups to consider are uniform lattices with torsion (which, by Selberg, are extensions of manifold groups by finite groups).

Theorem (Fowler)**.** *Let Γ be a uniform lattice containing an element of finite order ($\neq 2$). Although Γ is $\mathbb{Q}$-PD, Γ does not act freely on a $\mathbb{Q}$-acyclic $\mathbb{Q}$-homology manifold.*

The proof involves two obstructions: a finiteness obstruction, and a controlled symmetric signature. With more work, I hope to address the situation for Γ containing 2-torsion

Sketch of Proof: The group Γ is given as an extension $0 \to \pi \to \Gamma \to G \to 0$, where π is torsion-free, and G is a finite group. Given the Baum-Connes conjecture for Γ, the Γ-equivariant signature operator on $E\pi$ (which is the universal space for proper Γ actions) cannot be lifted back to the signature operator on $B\Gamma$.

To obstruct this lift, the space $E\pi$ is used to determine the equivariant signature operator of Γ acting on the universal cover of the hypothetical homology manifold; geometric localization arguments can then be combined with calculations near fixed points to give a contradiction—although this last step fails in the absence of odd torsion. $\square$

In addition to controlled surgery considerations, there is a finiteness question:

Question 3. For a given group G, does there exist a free action of G on an R-acyclic complex, with finite quotient?

Bestvina and Brady [**MR1465330**] call such groups FH over R, in analogy with other homological finiteness conditions.

In many cases, a finiteness obstruction will prevent Γ from acting on a $\mathbb{Q}$-acyclic finite complex, let alone a $\mathbb{Q}$-homology manifold. Again, consider a uniform lattice Γ, and write $0 \to \pi \to \Gamma \to G \to 0$, with π torsion-free and G finite. For convenience, $G = \mathbb{Z}/p\mathbb{Z}$. If G acts freely on a finite complex $B\pi$, then the Lefschetz fixed point theorem implies $\chi(B\pi^G) = 0$.

The equivariant finiteness obstruction fits into a categorical framework defined by Lück [**MR1027600**]; the condition $\chi(B\pi^G) = 0$, while necessary for Γ to be FH, is not enough: $\chi(B\pi^G)$ can vanish while the Euler characteristics of the connected components of $B\pi^G$ may not vanish. If $\chi(C) = 0$ for each connected component C of $B\pi^G$, then the equivariant finiteness obstruction does indeed vanish. The componentwise vanishing of the Euler characteristic is easy to verify in some cases, such as when the fixed sets are all odd-dimensional submanifolds.

1.2. **Aspherical manifolds and triangulations.** Kirby and Siebenmann showed that there are manifolds that do not admit PL structures [**MR242166**], and yet the possibility remained that all manifolds could be triangulated, meaning that for every manifold M, there is a simplicial complex K so that the geometric realization of K is homeomorphic to M, but of course the simplicial complex K is not a PL triangulation, meaning the links are not spheres.

Freedman showed that there are 4-manifolds that cannot be triangulated [**MR679066**]. Davis and Januszkiewicz applied a hyperbolization procedure to Freedman's 4-manifolds to get closed aspherical 4-manifolds that cannot be triangulated [**MR1131435**]. What about higher dimensions?

In the late 1970s Galewski and Stern [**MR558395**] and independently, Matumoto, showed that non-triangulable manifolds exist in all dimensions > 4 if and only if homology 3-spheres with certain properties do not exist. In [**manolescu2013conley**], Manolescu showed that there were no such homology 3-spheres, and hence non-triangulable manifolds exist in every dimension > 4.

Applying a hyperbolization technique to the Galewski-Stern manifolds shows the following.

Theorem 4 (Davis–Lafont–Fowler [**aspherical**])**.** *Let $n > 5$. There exists a closed aspherical n-manifold which cannot be triangulated.*

However, the following question remains open.

Question 5. Do there exist closed aspherical 5-manifolds that cannot be triangulated?

Computer search for Poincaré duality groups. There are well-known algorithms for computing the group cohomology of finite groups [**MR603653**], but there are also algorithms for computing the group cohomology of automatic groups [**MR2093885**], which I could improve to compute the cup product structure. The examples of $\mathbb{Q}$-PD groups in Section 1.1 have a common structure as extensions of finite (0-dimensional) groups.

Question 6. Can one find new examples of $\mathbb{Q}$-PD groups among automatic groups?

Considering that it is not yet clear what sorts of restrictions automaticity might impose on a Poincaré duality group, such examples are even interesting independent of their usefulness for understanding $\mathbb{Q}$-homology manifolds.

With the goal of generalizing the theory of automatic groups, Ranicki posed the problem of extending automatic structures on groups to a similar structure on group rings and forms over them [**MR1388296**]; developing a theory of "automatic algebra" is a great goal in its own right, and, practically speaking, such a theory would permit experimentation on algebraic Poincaré complexes using computer algebra systems.

A computer search could also proceed for new hyperbolic $\mathbb{Z}$-PD groups; for such Γ, it is now known that $B\Gamma$ is homotopy equivalent to a $\mathbb{Z}$-homology manifold (if $\dim > 4$, by combining [**MR1394965**] with work of Bartels and Lück). Upon finding some examples, it might then be possible to discover a group Γ so that $B\Gamma$ has the homotopy type of a $\mathbb{Z}$-homology manifold, but not the homotopy type of a manifold.

2. Computations surrounding high-dimensional manifolds

2.1. **Computing the L-polynomial.**

The Hirzebruch L-polynomial is one place where number theory very strongly interacts with high-dimensional topology [**MR339202**]. Recall that the Hirzebruch signature theorem relates the signature of a smooth closed manifold M^{4k} to $\sum_I L_I p_I(M)$. Unfortunately, the naïve method to compute coefficients L_I of the Hirzebruch L-polynomial is much too slow for applications; Zhixu Su and I have discovered a recursive method which is fast enough to compute many coefficients.

Solutions to some Diophantine equations related to these L-polynomials give rise to manifolds having a truncated polynomial algebra as their rational cohomology ring; such manifolds may exist even when the corresponding truncated polynomial algebra over $\mathbb{Z}$ is not the cohomology ring of any space. For instance, there is a manifold having the rational cohomology that $\mathbb{O}P^4$ would be expected to have, if $\mathbb{O}P^4$ existed.

On the other hand, nonexistence results are also possible. A rational projective plane means a smooth manifold M^{4n} with

$$H^\star(M;\mathbb{Q}) = \mathbb{Q}[x]/(x^3) \text{ with } |x| = 2n.$$

Question 7. For which n is there a rational projective plane M^{4n}?

There is a piecewise linear $\mathbb{Q}$-homology manifold with $\mathbb{Q}[x]/(x^3)$ as its cohomology ring, so this question can be viewed as a question about whether that $\mathbb{Q}$-homology manifold can be "resolved" by a smooth manifold. Integrally, this is not always possible by the celebrated work of Adams [**MR133837**], but in Zhixu Su's thesis, it was shown that there *is* a rational projective plane in dimension 32. But it is not always possible, even rationally.

Theorem 8 (Fowler–Su)**.** *There is no rational projective plane in dimension 64.*

This boils down to some number theory. Since a rational projective plane has signature one, we would be seeking a solution to $s_{8,8}x^2 \pm s_{16}y = \pm 1$ for integers x and y. Note that 37 divides the numerator of s_{16}, because 37 divides B_{32}—perhaps not so surprising considering 37's status as the smallest irregular prime. So it is enough to show there is no solution to $x^2 \not\equiv \pm 1/s_{8,8} \pmod{37}$. Since $s_{16} \equiv 0 \pmod{37}$,

$$\begin{aligned} s_{8,8} &\equiv \frac{{s_k}^2 - s_{2k}}{2} \pmod{37} \\ &\equiv \frac{{s_k}^2}{2} \pmod{37}, \end{aligned}$$

but neither 2 nor -2 is a quadratic residue modulo 37.

The same argument works to rule out a rational projective plane in dimension 2^{k+3} provided one can find a prime p so that

- 2 and -2 are quadratic nonresidues modulo p,
- $\nu_p(s_{2\cdot 2^k}) > 0$, but
- $\nu_p(s_{2^k}) = 0$.

To ensure 2 is a quadratic nonresidue, it is enough that $p \not\equiv \pm 1 \pmod 8$; to ensure that -2 is also a quadratic nonresidue, we further want $p \equiv 5 \pmod 8$.

The fact that divisors of $2^{2^k-1} - 1$ are rarely (never?) 5 mod 8 tells us not to look for such primes among the divisors of the Mersenne factor. The above desiderata are satisfied by finding a prime p so that

- $p \equiv 5 \pmod 8$,
- $p > 4 \cdot 2^k$,
- p divides the numerator of $B_{4\cdot 2^k}$,
- p does not divide the numerator of $B_{2\cdot 2^k}$.

The number theory becomes rather involved computationally. For example, the prime $p = 502261$ is 5 mod 8 and divides the numerator of $B_{4\cdot 2^{11}}$ but not $B_{2\cdot 2^{11}}$, which rules out a rational projective plane in dimension $2^{14} = 4096$; similarly, the prime $p = 69399493$ is 5 mod 8, divides the numerator of $B_{4\cdot 2^{21}}$ but not $B_{2\cdot 2^{21}}$, which rules out a projective plane in dimension 2^{24}. These calculations are possible due to tables of irregular primes produced by Joe P. Buhler and David Harvey [**MR2813369**].

Looking over these calculations, there are things that we can say in general.

Theorem 9 (Fowler–Su)**.** *If M^n is a rational projective plane, then $n = 2^a + 2^b$ for $a, b \in \mathbb{N}$.*

This result has some interesting consequences, such as the fact that there is a topological manifold which is not $\mathbb{Q}$-homotopy equivalent to a smooth manifold. And even in dimensions $4n$ for which there is a rational projective plane, a refined question can be explored.

Question 10. Suppose M^{4n} is a rational projective plane. How highly connected can M^{4n} be, integrally?

When $4n = 32$, we have a particularly nice answer by doing $\hat{A}$-genus calculations: there does not exist a simply-connected closed Spin manifold M^{32} which is a rational projective plane, so a 32-dimensional example cannot even be integrally 2-connected. Calculations involving the Steenrod algebra and Stiefel-Whitney classes may provide other methods for determining how highly connected a rational projective plane must be.

Finally, another interesting question is to study the asymptotic running time for algorithms computing the L-polynomial.

Question 11. How quickly can the L-polynomial be computed?

Such questions tie into Bernoulli number calculations, for which there are impressively fast analytic methods [**MR2684369**]. This would be a nice historical story, since the computation of Bernoulli numbers was the goal of a computer program from 1843 written by Ada Lovelace [**MR550674**].

2.2. **Computing L-classes.** Beyond the L-polynomial, computation of the total L-class would also permit certain manifolds to be recognized. Here is an example of such a problem and a possible approach. Brehm and Kühnel in [**MR1180457**] exhibit a few different combinatorial 15-vertex triangulations of an 8-manifold "like" the quaternionic projective plane $\mathbb{H}P^8$. One of these examples X^8 is especially symmetric, and likely PL homeomorphic to $\mathbb{H}P^8$.

Question 12. Is there a PL homeomorphism between the 15-vertex complex of Brehm–Kühnel and $\mathbb{H}P^8$?

Despite more recent work (e.g., [**MR3038783**]) which has placed these examples in a nice context, this question remains open. I propose answering this question with a direct computation of the rational L-class by implementing the procedure in [**MR440554**]. It is perhaps surprising that this can be done effectively. The relevant steps are to

- find a simplicial map $f : X^8 \times S^n \to S^{n+8}$ of nonzero degree,
- consider the preimage $f^{-1}(x)$ of some point $x \in S^{n+8}$, and
- compute the signature of $f^{-1}(x)$ by computing the cup product pairing on $H^\star(f^{-1}(x); \mathbb{Q})$.

Of these, finding the map f has proven to be more involved than I would have hoped; the stabilized copy of X^8 needs to be subdivided to get a map to a sphere, and this subdivision quickly increases the number of simplexes that need to be stored, in spite of how small the 15-vertex triangulation is at first. Nevertheless, I am optimistic that this is possible.

2.3. **Computing Stiefel-Whitney classes.** Having considered experiments with the L-classes—which amount to experiments with the Pontrjagin classes—there are also experiments one can do with Stiefel-Whitney classes. This project is joint work in progress with Jean Lafont.

Let G be a finite group. The Vasquez invariant $n(G)$ is the natural number so that, for any flat manifold M with holonomy G, then M is a toral expansion of some flat manifold with dimension $\leq n(G)$. Computing $n(G)$ is Problem 4 of [**MR2252897**].

That such a number $n(G)$ exists is a result of Vasquez' [**MR267487**], where Vasquez also points out that, in dimension $> n(G)$, the characteristic algebra of M vanishes. In other words, for a flat manifold M with holonomy G,

$$w_{i_1}(M) w_{i_2}(M) \cdots w_{i_n}(M) = 0$$

when $i_1 + i_2 + \cdots + i_n > n(G)$. For a given group G, by explicitly producing flat manifolds M with holonomy G, a nonzero product of Stiefel-Whitney classes yields a lower bound on $n(G)$. So how is one to produce a "random" flat manifold with holonomy G? Restrict to the case of $G = (\mathbb{Z}/2\mathbb{Z})^k$. Then a nice reasonable collection of flat manifolds are the real Bott manifolds, which are given by some combinatorial information [**MR2915482**], namely a matrix with entries in $\{0, 1\}$.

Question 13. Let $G = (\mathbb{Z}/2\mathbb{Z})^k$ and let M be a real Bott manifold with holonomy G. What is the largest nonzero product of Stiefel-Whitney classes of M?

Given a random matrix, I have already written code to compute the Stiefel-Whitney classes. Probably more can be said by using the fact that the holonomy group we are dealing with is abelian.

It would also be interesting to have some of explanation of the statistics I am seeing in the data. For example, with 26831 random 12-by-12 upper triangular matrices so that $w_2 w_3 w_4$ is nontrivial, only forty were found with holonomy $(\mathbb{Z}/2\mathbb{Z})^6$ but 10999 were found with holonomy $(\mathbb{Z}/2\mathbb{Z})^9$.

2.4. **Polynomially bounded homotopy theory.** In controlled topology, notions like homotopy are refined to include a condition on their size, measured via a reference map to a metric space. There are different versions of controlled topology in current use, including bounded control and continuous control. But there are other versions of control that are worth considering.

That is not so surprisingly: much of the success of geometric group theory comes by thinking about asymptotic invariants of a group as a metric space [**MR1253544**], but one can consider other asymptotic invariants for spaces.

Question 14. When does a CW complex have the homotopy type of a complex having polynomially many cells in dimension n?

This is a quantitative version of Wall's finiteness obstruction [**MR171284**]. Interestingly, a space might have Poincaré duality but not in a polynomially bounded sense, so there are analogies to be made between this project and the homology manifold projects.

Crichton Ogle has developed polynomially bounded cohomology [**MR2109110**]. I have collaborated with Ogle to work through the foundations of polynomially bounded—and more generally, $\mathcal{B}$-bounded—homotopy theory [**MR2962981**], which would be one framework within which the above question could be addressed. Specifically, given a bounding class $\mathcal{B}$, we construct a bounded refinement $\mathcal{B}K(-)$ of Quillen's K-theory functor from rings to spaces. As defined, $\mathcal{B}K(-)$ is a functor from weighted rings to spaces, and is equipped with a comparison map $BK \to K$ induced by "forgetting control." In contrast to the situation with $\mathcal{B}$-bounded cohomology, there is a functorial splitting $\mathcal{B}K(-) \simeq K(-) \times \mathcal{B}K^{rel}(-)$ where $\mathcal{B}K^{rel}(-)$ is the homotopy fiber of the comparison map.

Finally, there is also an analogy to be made between "weighted rings" and the geometric modules of Quinn [**MR802791**], along with $\mathcal{B}$-bounded homotopy theory and controlled topology. Much of our work in [**MR2962981**] is in building an appropriate $\mathcal{B}$-bounded Waldhausen category. At the point where one can define $\mathcal{B}$-bounded Waldhausen categories, why not go all the way and consider a $\mathcal{B}$-bounded model theory?

Christensen and Munkholm have placed continuous and bounded notions of control into a common, categorical framework [**MR1983017**]; Higson–Pedersen–Roe have introduced a different framework unifying various kinds of coarse spaces, from an analytic perspective [**MR1451755**]. Weiss–Williams formulate some examples of control within Waldhausen categories with duality [**MR1644309**]. All of these notions could be unified into a single theory of *controlled model categories*. Anderson's homotopy theory for boundedly controlled topology [**MR953961**] is a starting point.

There is some precedent for considering a controlled model category, namely the parametrized homotopy theory of May–Sigurdsson [**MR2271789**]. This latter theory, however, is more topological than geometric—the reference maps are not to metric spaces. Nevertheless, elucidating the precise relationship between the parametrized homotopy theory of May–Sigurdsson and the spectral cosheaves used in stratified surgery (see [**MR1308714**]) would be very worthwhile.

3. Undergraduate research

There are two different ways to involve undergraduates in research. One way is to do research projects *with* undergraduates: I have two successful projects that involved undergraduates in a significant way, as well as some proposed projects that I hope to do with future students. The other way to involve undergraduates is to do research *on* them: I have an ongoing project called MOOCulus, an adaptive learning platform that I built. There have been about 150k enrollments in this online course, so we have quite a bit of data.

3.1. **No three in line.** For a group G, let $T(G)$ denote the cardinality of the largest subset $S \subset G$ so that no three elements of S are in the same coset of a cyclic subgroup. Undergraduates Andrew Groot and Deven Pandya, advised by myself and my colleague Bart Snapp, considered the case $G = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, and showed that

$$T(\mathbb{Z}_p \times \mathbb{Z}_{p^2}) = 2p,$$
$$T(\mathbb{Z}_p \times \mathbb{Z}_{pq}) = p + 1.$$

This problem can also be formulated as a Gröbner basis question; after doing so, we computed $T(\mathbb{Z}_m \times \mathbb{Z}_n)$ for $2 \le m \le 7$ and $2 \le n \le 19$. These results are available in [**2012arXiv1203.6604F**].

This problem presents some nice connections for undergraduates. Thinking of a coset of a cyclic subgroup as a "line," there is then connection with the usual "no three in line problem." Paul Erdös proved that for a prime p, one can place p points on the $p \times p$ lattice in the plane [**MR41889**]; the construction goes via a parabola modulo p. Other more complicated constructions manage to place more points [**MR366817**].

Another nice connection is made by considering other groups. Although the no-three-in-line problem for $G = (\mathbb{Z}/p\mathbb{Z})^2$ can be considered as the k-arc problem from projective geometry [**MR554919**], the question is also interesting for, say, $G = S_n$ or $G = A_n$ where, say, Bezout's theorem doesn't make sense anymore.

3.2. **Transversal of primes.** Matej Penciak, then a student at the University of Rochester, among other students at the Ross program, became interested in the following question.

Question 15 (Question 1.6 "Transversal of Primes" from [**MR2847943**])**.** Let p be the n^{th} prime, and place the integers 1 through p^2 into a $p \times p$ array in order. Is it possible to choose a set of p primes from the array so that no two appear in the same row or column?

During Summer 2012, we gathered some evidence on this question. Our approach used WalkSAT [**walksat**], a local search algorithm for Boolean satisfiability problems. We verified that it is possible to arrange p primes on a $p \times p$ board, no two in a row or column, for primes ≤ 1291.

3.3. **Proposed undergraduate research projects.** Here are some proposed projects that could become good activities for undergraduates.

Space of five-by-five natural images: Carlsson–Ishkhanov–De Silva–Zomorodian found a Klein bottle in the space of 3×3 pixel patches of natural images [**natural-images**]. To be a bit more precise, each 3×3 pixel patch from a black-and-white image yields a point in $[0,1]^9 \subset \mathbb{R}^9$; one can normalize for brightness and contrast, and then apply a "thresholding" procedure to throw away some outliers. Many of the remaining points are close to a Klein bottle embedded in $\mathbb{R}^9$. A nice project would be describe the structure of, say, 5×5 pixel patches. A first step would be to compute the persistent homology [**MR2121296**] of the (suitably "thresholded") point cloud, which could be done with available software by an undergraduate having some background in computer science.

Roots of random quaternionic polynomials: There are other examples of point cloud data coming from "pure mathematics" for which it would be interesting to calculate the persistent homology. For example, there are well-known results on the distribution of complex roots of random polynomials, some of which are topologically interesting—for some distributions, the roots lie close to a circle. There are algorithms for finding roots of a quaternionic polynomial [**MR6980**] which can be performed in practice [**MR1851239**], so what sort of topological structure might one find in the roots of random quaternionic polynomials? The same skillset that would help an undergraduate analyze the space of 5×5 pixel patches could be applied to this question, as well.

Representation theory and the genetic code: The genetic code describes how sequences of DNA are transformed into proteins; the code is redundant, but why? Considering that there are a large number of genetic codes, why should we see only a handful of possible codes in nature? Hornos and Hornos investigated the symmetries of the genetic code via representation theory [**originalhornos**], akin to the techniques used in particle physics with Lie groups. But the genetic code is discrete, so instead of the representation theory of Lie groups, *modular* representation theory is arguably the natural mathematical home for such study. The "wobble pair" phenomena might be suggestive that symplectic geometry is relevant, i.e., thinking of a codon—three base pairs—not as a vector in $(\mathbb{F}_4)^3$ but as a vector in $(\mathbb{F}_2)^6$ adorned with a symplectic form. I found a 64-dimensional (i.e., $64 = 4^3$ codons) representation of $\mathrm{Sp}(6, \mathbb{F}_2)$ over the field $\mathbb{F}_2$ in which there are 21 minimal submodules for a smaller symmetry group (i.e., 20 amino acids and 1 stop code); moreover, the dimensions of these minimal submodules are just large enough to account for the redundancy in the genetic code.

It would be interesting to work with, say, an undergraduate student interested in math biology, teach them a bit of representation theory, and work out some of the consequences. Of course, this project, like most of the work done on representation theory and the genetic code, is highly speculative. One complaint is that such work is "mere numerology" which does not even succeed in giving the correct answer for the standard genetic code: for instance, there is no example in [**forger-1999**, **forger-1998**] of a Lie group having precisely the correct representation theory to give rise to the standard genetic code, though there is a near-miss differing only in the symmetry breaking at the last stage.

3.4. **MOOCs and adaptive learning.** In January 2013, I launched the first massive open online course (MOOC) at The Ohio State University. My course was designed to cover the same content as the local, in-person sections of Math 1151, the first semester calculus course at OSU. The largest component of this MOOC consisted of a home-built adaptive learning platform I designed, called MOOCulus, which delivers randomly-generated interactive problems to students [**evans**].

Beyond my $\sim$ 200 lecture videos—some of which make use of augmented reality—the MOOCulus platform has generated significant amounts of data on the learning of calculus. As of November 2013, we have had 147k students enroll in either the Coursera or iTunes U version of my courses. These enrollments have led to millions of attempts, and over two million correct answers, being submitted to MOOCulus.

The Division of Undergraduate Education at the NSF has funded a Transforming Undergraduate Education in STEM (TUES) Type 1 award for my proposal DUE–1245433 ("Interactive Textbook") with Bart Snapp and Herb Clemens. The proposed activity will produce an online platform making it easier for other instructors to build engaging online courses like MOOCulus. The goal is to build one of the "customizable, sustainable platforms" that online education requires to succeed [**bowen2013higher**]; parts of MOOCulus have already been used for English courses as well [**gates-foundation-grant**].

Much previous work has been done on using adaptive learning systems to teach college mathematics. Examples of such are ALEKS [**hagerty2005using**, **albert1999knowledge**] or Knewton at Arizona State [**parry2012big**], though both of these emphasize pre-calculus, developmental coursework. Some of these platforms use Bayesian networks to estimate a student's current understanding [**romero2010educational**]. Hidden Markov models (HMMs) have also been used in educational data mining for clustering [**shihdiscovery**]. These techniques and related "machine learning" techniques can predict student grades based on only a few assignments [**predict-grades**], even with incomplete data [**Zafra201115020**], which make them especially useful for an adaptive learning platform where not too much student data may be available on which to base the initial predictions.

For MOOCulus, I chose to use a hidden Markov model to estimate student understanding; exactly which problems are assigned to the student depend on the output of the model. Hidden Markov models can provide strong predictions of student behavior; in one example, a hidden Markov model—trained, as I propose here, via the Baum–Welch algorithm—did a better job of predicting student behavior in an e-Learning context than did a neural net [**anari2012intelligent**]. Nevertheless, exactly how best to train the MOOCulus model remains a question.

Question 16. What are the "correct" parameters for the hidden Markov model in MOOCulus?

Parameter estimation is an extremely well-studied topic in the theory of Markov processes, e.g., [**MR202264**] and [**MR123419**]. Before having access to student data, instructors often "have to provide appropriate values for the parameters in advance in order to obtain good results/model and therefore, the user must possess a certain amount of expertise in order to find the right settings" [**romero2010educational**]. This is a serious usability problem with many adaptive learning systems—MOOCulus included.

Determining appropriate values for these parameters is a usability issue, and also one of the major research questions my work addresses. The plan is to use the Baum–Welch algorithm from the GHMM library [**ghmm**] so improvements to the parameters can be made automatically. The overarching goal is that MOOCulus will improve student outcomes in math courses, as has been shown to happen with other online assessment systems [**angus2009does**].