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ABSTRACT

There exist torsion-free finitely presented groups satisfying R-Poincaré duality, which are nonetheless not the fundamental group of a closed aspherical R-homology manifold (answering a question posed in [Dav00]); the construction combines Bestvina–Brady Morse theory with an acyclic variant of M. Davis' reflection group trick.

That this is possible suggests replacing *aspherical* with *acyclic universal cover*: is every finitely presented R-Poincaré duality group the fundamental group of a R-homology manifold with R-acyclic universal cover? This question can be asked even for groups containing torsion; we construct examples of such groups for all $\mathbb{Z} \subsetneq R \subseteq \mathbb{Q}$.

However, we also show that this is rather exceptional: uniform lattices in semisimple Lie groups which contain p-torsion (for $p \neq 2$) do not act freely on $\mathbb{Q}$-acyclic $\mathbb{Q}$-homology manifolds; obstructions include an equivariant finiteness obstruction and a lifting problem for rational controlled symmetric signatures.

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And God has blessed me through my wife, Jana. When Proverbs 31 asks who can find an excellent wife, Proverbs 19:14 says that such a wife is from God. Jana has supported me through difficult times with wisdom and with prayer, and she has made my joyful times happier than I would have thought possible. I love her, and I dedicate this dissertation to her.

James Fowler
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CHAPTER 1
INTRODUCTION

1.1 Historical Context

Borel's famous conjecture asserts that aspherical manifolds are topologically rigid.

Conjecture 1.1.1 (Borel). Suppose $f : M \to N$ is a homotopy equivalence between closed aspherical manifolds; then f is homotopic to a homeomorphism.

Given the conjectured uniqueness of aspherical manifolds, an existence question remains: which groups π are fundamental groups of closed aspherical manifolds? A necessary condition is that $K(\pi, 1)$ satisfy Poincaré duality (in which case we call π a "Poincaré duality group" [JW72]). Wall asked whether this suffices.

Question 1.1.2 (Wall in [Wal79], problem G2, page 391). Is every Poincaré duality group Γ the fundamental group of a closed $K(\Gamma, 1)$ manifold? Smooth manifold? Manifold unique up to homeomorphism?

The history of this problem is discussed in [FRR95a], [FRR95b], and [Dav00].

Bryant–Ferry–Mio–Weinberger have developed a surgery theory for ANR $\mathbb{Z}$-homology manifolds [BFMW96, BFMW93]; compared to surgery theory for topological manifolds, surgery theory for ANR $\mathbb{Z}$-homology manifolds has an improved Siebenmann periodicity—one hint that ANR $\mathbb{Z}$-homology manifolds are more basic objects than topological manifolds.

Conjecture 1.1.3 (Bryant–Ferry–Mio–Weinberger). If a finitely presented group π satisfies Poincaré duality, then π is the fundamental group of an aspherical closed ANR $\mathbb{Z}$-homology manifold.

Their work shows that this conjecture follows from (an algebraic restatement of) the Borel conjecture. For instance, recent work of Bartels and Lück [BL09] implies Conjecture 1.1.3 when Γ is hyperbolic and dim > 5, while Wall's original question can be affirmed only under more restrictive conditions on the Gromov boundary (see [Gro81]).

1.2 Main Question

The existence and uniqueness questions for closed aspherical $\mathbb{Z}$-homology manifolds can be formulated for R-homology manifolds. Mike Davis does this in [Dav00]; his question asks if some algebra (i.e., having R-Poincaré duality) is necessarily a consequence of some geometry (i.e., being an R-homology manifold).

Question 1.2.1 (M. Davis). Is every torsion-free finitely presented group satisfying R-Poincaré duality the fundamental group of an aspherical closed R-homology n-manifold?

We prove in Chapter 3 that the answer to the above question is no. However, the construction of the counterexample, as well as the spirit of the original question, suggests weakening the conclusion.

Question 1.2.2 (Acyclic variant of a question of M. Davis). Suppose Γ is a finitely presented group satisfying R-Poincaré duality. Is there a closed R-homology manifold M, with

- $\pi_1 M = \Gamma$, and
- $H_\star(\tilde{M}; R) = H_\star(\bullet; R)$, in other words, R-acyclic universal cover?

Instead of asking for an aspherical homology manifold (as in M. Davis' original question), this modified question only asks that the homology manifold have R-acyclic universal cover. Nevertheless, a group acting geometrically and cocompactly on an R-acyclic R-homology manifold still possesses R-Poincaré duality, so the setup in Question 1.2.2 provides a "geometric source" for the R-Poincaré duality of a group.

1.3 Results

By asking for an R-acyclic universal cover, we permit the possibility that the group contains torsion. In Chapter 4, we will construct examples illustrating this possibility.

Theorem 1.3.1. *Let $G = \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Then there exists a $\mathbb{Q}$-homology manifold M so that $\pi_1 M$ retracts onto G, and $\tilde{M}$ is $\mathbb{Q}$-acyclic.*

However, in Chapter 5, we show that, for many groups, such $\mathbb{Q}$-homology manifolds do not exist.

Theorem 1.3.2. *Let Γ be a uniform lattice in a semisimple Lie group containing p-torsion (for $p \neq 2$). There does not exist an ANR $\mathbb{Q}$-homology manifold X, with $\pi_1 X = \Gamma$, and $\mathbb{Q}$-acyclic universal cover $\tilde{X}$.*

The theorem answers Question 1.2.2 in the negative. Ignoring an orientation issue, these groups Γ are virtually $\mathbb{Z}$-Poincaré duality groups (in fact, they are virtually the fundamental groups of aspherical closed manifolds); such a "virtual manifold group" satisfies $\mathbb{Q}$-Poincaré duality, but, by Theorem 1.3.2, need not be the fundamental group of a $\mathbb{Q}$-homology manifold with $\mathbb{Q}$-acyclic universal cover.

This can also be looked at from the perspective of orbifolds: the locally symmetric space $K\backslash G/\Gamma$ is an orbifold, and the "orbifold fundamental group" is Γ, but what about the usual fundamental group $\pi_1(\Gamma\backslash G/K)$? Such Γ cannot be the fundamental group of $K\backslash G/\Gamma$, or any ANR $\mathbb{Q}$-homology manifold with $\mathbb{Q}$-acyclic universal cover.

If a group satisfying $\mathbb{Q}$-Poincaré duality has any hope of being the fundamental group of a compact ANR $\mathbb{Q}$-homology manifold, it must satisfy a particular finiteness property: specifically, the group must be the fundamental group of a compact space with $\mathbb{Q}$-acyclic universal cover. In Chapter 4, we study this question in greater generality.

Theorem 1.3.3. *Suppose*

$$1 \to \pi \to \Gamma \to G \to 1$$

with $B\pi$ a finite complex and G a finite group; the group G acts on $B\pi$, and if, for all nontrivial subgroups $H \subset G$, and every connected component C of the fixed set $(B\pi)^H$,

$$\chi(C) = 0,$$

then there exists a compact space X with $\pi_1 X = \Gamma$ and $\tilde{X}$ rationally acyclic.

In addition to the sufficient condition stated in this theorem, a necessary condition is that $\chi\left((B\pi)^{\langle g \rangle}\right) = 0$ for all nontrivial cyclic subgroups $\langle g \rangle \subset G$. The finiteness question can be studied even for nonuniform lattices Γ.

The proof of Theorem 1.3.3 uses the equivariant finiteness theory of W. Lück [Lüc89], but we recast it in somewhat different language (akin to [DL98, DL03]), making it more obviously functorial. In any case, the hypotheses of the Theorem 1.3.3 are not hard to satisfy: for instance, by crossing with S^1, or by guaranteeing that the fixed sets are all odd-dimensional manifolds, and therefore have vanishing Euler characteristic. Producing a $\mathbb{Q}$-homology manifold (as in Theorem 1.3.2) is further obstructed by higher signatures, which will not vanish even after crossing with S^1.

CHAPTER 2
PRELIMINARIES

Throughout, R-homology manifold means ANR R-homology manifold.

2.1 Finiteness properties

There are many properties which measure the "finiteness" of a(n ironically often infinite) group; there are many such properties, which have been given short names like F, FP, etc. A good reference is [Bro82].

The first finiteness properties we consider are topological.

Definition 2.1.1. Let Γ be a group; if $B\Gamma$ has the homotopy type of a complex K with its n-skeleton $K^{(n)}$ a finite complex, we say Γ has property F_n; in the case where $B\Gamma$ has the homotopy type of a finite complex, we say Γ has property F.

For convenience, we confuse these properties with the classes of groups satisfying them, allowing us to write "$\Gamma \in \mathrm{F}$" to mean that Γ satisfies F. Note that F is a proper subclass of $\bigcap_{n\in\mathbb{N}} \mathrm{F}_n$. Additionally, some of the F_n are equivalent to more familiar properties.

Proposition 2.1.2. *A group G is F_1 if and only if G is finitely generated.*
A group G is F_2 if and only if G is finitely presented.

Property FH, introduced in [BB97], is a homological variant of the homotopical property F.

Definition 2.1.3. Let Γ be a group; if Γ acts freely, faithfully, properly discontinuously, cellularly, and cocompactly on a cell complex X with

$$H_k(X;R) = H_k(\bullet;R) \text{ for } 0 \leq k \leq n-1,$$

then we say Γ has property $\mathrm{FH}_n(R)$. If Γ acts as above on an R-acyclic complex X, then we say Γ has property $\mathrm{FH}(R)$.

Complementing these topological finiteness conditions, there are other finiteness properties which are more algebraic.

Definition 2.1.4. Let Γ be a group; if the trivial $R\Gamma$-module R admits a resolution

$$M_n \to \cdots \to M_0 \to R$$

with the M_i finitely generated projective $R\Gamma$-modules, we say that Γ has property FP_n. If the trivial $R\Gamma$-module R admits a finite length resolution by finitely generated projective $R\Gamma$-modules, we say that Γ has property FP.

As for F, note that FP is a proper subclass of $\bigcap_{n\in\mathbb{N}} \mathrm{FP}_n$.

Analogous to property FP, there is a property FL, satisfied by groups admitting finite length resolutions by finitely generated free modules.

Definition 2.1.5. Let Γ be a group; if the trivial $R\Gamma$-module R admits a resolution

$$M_n \to \cdots \to M_0 \to R$$

with the M_i finitely generated free $R\Gamma$-modules, we say that Γ has property FL_n. If the trivial $R\Gamma$-module R admits a finite length resolution by finitely generated free $R\Gamma$-modules, we say that Γ has property FL.

2.2 Poincaré duality groups

Our main goal is to understand the difference between those groups which satisfy Poincaré duality, and those which are fundamental groups of certain kinds of manifolds.

Definition 2.2.1. A group Γ is an *n-dimensional R-Poincaré duality group*, written $\Gamma \in \mathrm{PD}_n(R)$, if

- Γ is $\mathrm{FP}(R)$,
- $H_n(B\Gamma; R\Gamma) = R$, and $H_k(B\Gamma; R\Gamma) = 0$ for $k \neq n$.

Some authors also consider *duality groups*, without the $\mathrm{FP}(R)$ condition, and permitting other "orientation modules" besides the ring iteslf [BE73]. K. Brown shows that, when $R = \mathbb{Z}$, $\mathrm{FP}(\mathbb{Z})$ is implied by the homological condition alone [Bro75].

It is important to realize that $\Gamma \in \mathrm{FP}(R)$ does not imply $\Gamma \in \mathrm{FH}(R)$; in other words, the definition of $\mathrm{PD}(R)$ includes an algebraic finiteness condition but not a geometric finiteness condition. However, Poincaré duality can also be phrased in more geometric language:

Proposition 2.2.2. *A group $\Gamma \in \mathrm{PD}_n(R)$ if and only if $B\Gamma$ is an R-Poincaré duality complex.*

2.3 Manifold groups

Traditioanlly, a group π is an "aspherical n-manifold group" means that $B\pi$ has the homotopy type of a closed n-manifold. We broaden this class to include the possibility of closed aspherical homology manifolds.

Definition 2.3.1. A group is an *aspherical R-homology n-manifold group* if it is the fundamental group of a closed aspherical R-homology n-manifold. If Γ is such a group, we write $\Gamma \in \mathrm{AsphMfld}_n(R)$.

Obviously, such a group has a finite classifying space, i.e.,

Proposition 2.3.2. $\mathrm{AsphMfld}_n(R) \subset \mathrm{F}$.

Using our notation, Davis' original question can be phrased as whether

$$\mathrm{AsphMfld}_n(R) = \mathrm{PD}_n(R) \cap \mathrm{F}_2 \cap \{\text{torsion-free groups}\}.$$

Asphericity is perhaps too restrictive; having R-acyclic universal cover seems more natural in the context of R-homology manifolds.

Definition 2.3.3. A group is an *R-homology n-manifold group* if it is the fundamental group of an R-homology n-manifold having R-acyclic universal cover. If Γ is such a group, we write $\Gamma \in \mathrm{Mfld}_n(R)$.

The following proposition is obvious.

Proposition 2.3.4. $\mathrm{Mfld}_n(R) \subset \mathrm{FH}(R)$.

Acting freely on an R-acyclic R-homology manifold suffices to satisfy R-Poincaré duality.

Proposition 2.3.5. $\mathrm{Mfld}_n(R) \subset \mathrm{PD}_n(R)$.

2.4 Relationships

The various finiteness properties and Poincaré duality and manifold properties are related; these relationships are summarized in a diagram.

$$\begin{array}{ccccccc}
\mathrm{AsphMfld} & \subset & \mathrm{Mfld} & & \subset & & \mathrm{PD} \\
\cap & & \cap & & & & \cap \\
\mathrm{F} & \subset & \mathrm{FH} & \subset & \mathrm{FL} & \subset & \mathrm{FP}
\end{array}$$

Observe that the hierarchy of finiteness properties is mirrored by the hierarchy of manifold properties. Whether these inclusions are sharp is an important question: for example, it is not known whether $\mathrm{FP}(\mathbb{Z}) = \mathrm{FL}(\mathbb{Z})$ or whether $\mathrm{Mfld}(\mathbb{Z}) = \mathrm{F}_2 \cap \mathrm{PD}(\mathbb{Z})$, or even whether $\mathrm{AsphMfld}_n(\mathbb{Z}) = \mathrm{Mfld}_n(\mathbb{Z})$.

Question 1.2.2 can be stated succintly as

Question 2.4.1. Does $\mathrm{Mfld}_n(R)$ equal $\mathrm{F}_2 \cap \mathrm{PD}_n(R)$?

This is too naïve: as we will see in chapter 4, differences between $\mathrm{FL}(R)$ and $\mathrm{FP}(R)$ can be detected via a finiteness obstruction. There are groups satisfying $\mathrm{PD}_n(R)$ which satisfy $\mathrm{FP}(R)$, but fail to satisfy $\mathrm{FL}(R)$, let alone $\mathrm{FH}(R)$.

What if we assume that the group satisfies $\mathrm{FH}(R)$? This is still insufficient to guarantee that the group satisfies $\mathrm{Mfld}(R)$.

Proposition 2.4.2. $\mathrm{Mfld}_n(R)$ *is a proper subclass of* $\mathrm{F}_2 \cap \mathrm{PD}_n(R) \cap \mathrm{FH}(R)$ *for* $R = \mathbb{Q}$.

Specifically, the material in chapter 5 will exhibit a group which is finitely presented, $\mathrm{PD}_n(\mathbb{Q})$ and $\mathrm{FH}(\mathbb{Q})$, but not $\mathrm{Mfld}_n(\mathbb{Q})$.

However, if we assume further that the group satisfies F, our methods no longer apply, so we cannot answer the following interesting questions.

Question 2.4.3. Does $\mathrm{AsphMfld}_n(R) = \mathrm{F} \cap \mathrm{PD}_n(R)$?

Question 2.4.4. Does $\mathrm{AsphMfld}_n(R) = \mathrm{F} \cap \mathrm{Mfld}_n(R)$?

2.5 Virtual properties

Often, it is worthwhile comparing a group to its finite index subgroups,

Definition 2.5.1. Let P be a class of groups (e.g., those groups satisfying a property P); the class VP consists of those groups which contain a finite index subgroup in P (e.g., those groups which *virtually* satisfy the property P).

For example, VF consists of those groups which contain a finite index subgroup H with $K(H,1)$ (homotopy equivalent to) a finite complex. Obviously $\mathrm{P} \subset \mathrm{VP}$, but in many cases the inclusion is proper.

If a group virtually satisfies a property over the ring $\mathbb{Z}$, the group might *actually* have the property over $\mathbb{Q}$. For example,

Proposition 2.5.2. $\mathrm{VPD}(\mathbb{Z}) \subset \mathrm{PD}(\mathbb{Q})$.

Proof. Proposition 4.5.5 implies that $\mathrm{VPD}(\mathbb{Z}) \subset \mathrm{PD}(\mathbb{Q})$.

$\square$

However, C.T.C. Wall proved in [JW72]

$$\mathrm{PD}(\mathbb{Z}) = \mathrm{VPD}(\mathbb{Z}) \cap \{\text{torsion-free groups}\}$$

Contrasting with the fact that $\mathrm{VPD}(\mathbb{Z}) \subset \mathrm{PD}(\mathbb{Q})$,

Proposition 2.5.3. $\mathrm{VFH}(\mathbb{Z}) \not\subset \mathrm{FH}(\mathbb{Q})$.

This follows from a finiteness obstruction, as we will see in Chapter 4.

Proposition 2.5.4. $\mathrm{VPD}(\mathbb{Q}) = \mathrm{PD}(\mathbb{Q})$, *but* $\mathrm{VMfld}(\mathbb{Q}) \neq \mathrm{Mfld}(\mathbb{Q})$.

In chapter 5, we will exhibit $\Gamma \in \mathrm{VMfld}(\mathbb{Z}) \subset \mathrm{VMfld}(\mathbb{Q})$ with $\Gamma \notin \mathrm{Mfld}(\mathbb{Q})$.

CHAPTER 3
TORSION-FREE EXAMPLE

3.1 Introduction

In [Dav98], M. Davis combined Bestvina–Brady Morse theory with the reflection group trick to produce Poincaré duality groups that are not finitely presented—and therefore, not fundamental groups of aspherical manifolds. With some care, this technique can be applied to rational Poincaré duality groups and rational homology manifolds, answering Question 1.2.1 in the negative.

Theorem 3.1.1. *There exists a torsion-free, finitely presented* $\mathrm{PD}(\mathbb{Q})$*-group* Γ *which is not the fundamental group of an aspherical closed* $\mathbb{Q}$*-homology manifold.*

The construction of such a group Γ proceeds as follows:

- Let X be a simply connected finite complex which is $\mathbb{Q}$-acyclic but not $\mathbb{Z}$-acyclic.
- Apply Bestvina–Brady Morse theory [BB97] to X; this produces a group $G \notin \mathrm{FP}(\mathbb{Z})$ with $G \in \mathrm{FH}(\mathbb{Q})$, so G acts freely and cocompactly on a $\mathbb{Q}$-acyclic space; let K be the quotient of such a free action.
- Apply a variant of M. Davis' reflection group trick [Dav83] to a thickened version of K; after taking a cover, this produces a torsion-free group Γ satisfying $\mathrm{PD}(\mathbb{Q})$.
- Verify that the Eilenberg-MacLane space $K(\Gamma, 1)$ is not homotopy equivalent to a finite complex.

Closed ANR $\mathbb{Q}$-homology manifolds are homotopy equivalent to finite complexes [Wes77]. Since $K(\Gamma, 1)$ is not homotopy equivalent to a finite complex, Γ cannot be the fundamental group of an aspherical closed $\mathbb{Q}$-homology manifold.

Briefly, Bestvina–Brady Morse theory proves that $\mathrm{F} \subset \mathrm{FH}(\mathbb{Q})$ is a strict inclusion, which will be promoted to a strict inclusion $\mathrm{AsphMfld}(\mathbb{Q}) \subset \mathrm{Mfld}(\mathbb{Q})$ via Davis' reflection group trick.

3.1.1 Related questions

Theorem 3.1.1 suggests varying the hypotheses or conclusion of Question 1.2.1.

On the one hand, we can strenghten the hypotheses. The group Γ we construct fails to be a homology manifold group because of a nonzero finiteness obstruction. What if we assume that the finiteness obstruction vanishes? Are there groups Γ which are $\mathrm{PD}(R)$ and with $K(\Gamma, 1)$ having the homotopy type of a finite complex, but which are not fundamental groups of aspherical R-homology manifolds? In other words, is $\mathrm{AsphMfld}(R) = \mathrm{F} \cap \mathrm{PD}(R)$? A related question is whether $\mathrm{AsphMfld}(R) = \mathrm{F} \cap \mathrm{Mfld}(R)$ holds.

On the other hand, we can weaken the conclusion of Question 1.2.1, from aspherical to acyclic; the construction, after all, produced an acyclic example. We ask: are there finitely presented groups which are $\mathrm{PD}(R)$ but nevertheless do not act freely on an R-acyclic R-homology manifold? Stated differently, is it the case that

$$\mathrm{Mfld}(R) = \mathrm{F}_2 \cap \mathrm{FH}(R) \cap \mathrm{PD}(R)?$$

Chapter 5 will prove this is not the case for $R = \mathbb{Q}$, but those examples are not torsion-free. The possibility that

$$\mathrm{Mfld}(R) = \mathrm{F}_2 \cap \mathrm{FH}(R) \cap \mathrm{PD}(R) \cap \{\text{torsion-free groups}\}$$

remains open, though it certainly seems unlikely.

3.2 Definitions

3.2.1 Groups from graphs

In using PL Morse theory and the reflection group trick, we will be making use (respectively) of Artin groups and Coxeter groups. These are families of groups associated to graphs with edges labelled by integers.

Definition 3.2.1. Let K be a 1-complex with vertices $v_1, \ldots, v_n$; the edge connecting v_i and v_j has a label $m_{ij} \in \mathbb{Z} \cup \{\infty\}$; if there is no edge connecting v_i and v_j, then we set $m_{ij} = \infty$. By convention, $m_{ii} = 1$.

The *Artin group* A_K associated to K has presentation

$$\langle g_1, \ldots, g_n | (g_i g_j)^{m_{ij}} = 1 \text{ for } 1 \leq i < j \leq n \rangle.$$

The *Coxeter group* C_K associated to K has presentation

$$\langle g_1, \ldots, g_n | (g_i g_j)^{m_{ij}} = 1 \text{ for } 1 \leq i \leq j \leq n \rangle.$$

In an Artin group, each g_i generates an infinite cyclic group; in a Coxeter group, each g_i generates $\mathbb{Z}/2$, since we include the relator $(g_i g_i)^1 = {g_i}^2 = 1$.

Definition 3.2.2. When all the edge labels m_{ij} for $i \neq j$ are either 2 or ∞, we call the resulting Coxeter (or Artin) group a *right-angled* Coxeter (or Artin) group.

Proposition 3.2.3. *An Artin group is torsion-free.*
A Coxeter group is virtually torsion-free.

3.2.2 Flag triangulations

Definition 3.2.4. A simplicial complex L is *flag* if, every inclusion of the 1-skeleton of an n-simplex in L,

$$\Delta_n{}^{(1)} \subset L,$$

factors through an n-simplex,

$$\Delta_n{}^{(1)} \subset \Delta_n \subset L.$$

Being flag does not restrict the topology of the complex L, by the following result.

Proposition 3.2.5. *If K is a simplicial complex, and the simplicial complex L is the barycentric subdivision of K, then L is flag.*

Flag complexes are determined by their 1-skeleton; a right-angled Coxeter group (or Artin group) can be built from this 1-skeleton (by giving each edge the label 2).

3.2.3 CAT(0) complexes

An excellent resource for CAT(0) complexes is the book by Bridson and Haefliger [BH99].

Definition 3.2.6. A path metric space X is CAT(0) if, for any geodesic triangle with vertices $a, b, c \in X$, the comparison triangle $a', b', c' \in \mathbb{R}^2$ is "thicker."

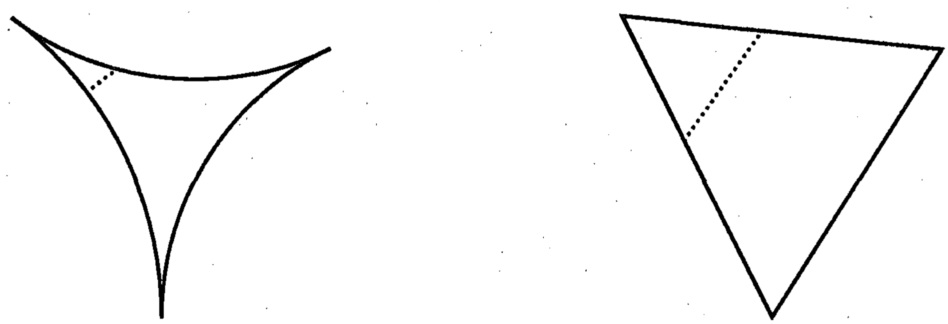
A *comparison triangle* is a triangle in $\mathbb{R}^2$ having the same side lengths; a triangle is *thicker* if corresponding arcs (i.e., starting and ending at points having equal distances from the vertices) are no shorter in $\mathbb{R}^2$ than in X.

This is illustrated in Figure 3.1.

Any CAT(0) space is contractible, so we are usually interested only in *locally* CAT(0) spaces, meaning that the CAT(0) inequality is satisfied for sufficiently small triangles. The most important feature of locally CAT(0) spaces is that, like Riemannian manifolds of nonpositive curvature, they are aspherical. Specifically, a simply connected locally CAT(0) space is globally CAT(0), and therefore, contractible.

Additionally, for cubical complexes, CAT(0) is equivalent to a checkable condition on the links of vertices. The link of a vertex in a cubical complex is a simplicial complex.

Figure 3.1: Comparison triangles for the CAT(0) inequality



Proposition 3.2.7. *A cubical complex (with the Euclidean metric on each cube) is locally* CAT(0) *if the link of each vertex is a flag simplicial complex (see Definition 3.2.4).*

The property CAT(-1), for combinatorial strict negative curvature, has a similar definition, except that the comparisons are made not to the Euclidean plane, but to the hyperbolic plane $\mathbb{H}^2$. We will not have much need for CAT(-1), but veryifying CAT(0) will be a useful way to produce aspherical spaces.

3.3 Construction

In this section, we prove Theorem 3.1.1; specifically, we exhibit a torsion-free, finitely presented PD($\mathbb{Q}$)-group which is not the fundamental group of an aspherical finite complex, let alone a closed ANR $\mathbb{Q}$-homology manifold.

3.3.1 PL Morse theory

We begin by producing a group which satisfies a rational finiteness property, but not an integral finiteness property. Choose a simply connected finite complex X

which is $\mathbb{Q}$-acyclic but not $\mathbb{Z}$-acyclic; for the sake of concreteness,

$$X = S^2 \cup_f e^3 \text{ with } f : \partial e^3 = S^2 \to S^2 \text{ degree two.}$$

Let L be a flag triangulation of X.

There is a construction that transforms a flag complex (e.g., L) into a CAT(0) cubical complex; basically, one considers the union of tori

$$X = \bigcup_{\sigma \in L} \prod_{v \in \sigma} S^1.$$

It is easy to check that the link of each vertex is L, and that, because L is flag, this means that X is CAT(0).

The following theorem summarizes a result of Bestvina–Brady PL Morse theory [BB97]; this is a version of Morse theory designed to analyze spaces such as the above cubical complex.

Theorem 3.3.1. *Let L be a finite flag complex. Let $A = A_L$ the associated right angled Artin group, and $G = G_L$ the kernel of a natural map $A_L \to \mathbb{Z}$.*

- *If L is R-acyclic, then $G \in \mathrm{FH}(R)$.*
- *If L is simply connected, then G is finitely presented.*
- *If L is not R-acyclic, then $G \notin \mathrm{FP}(R)$.*

Let $G = G_L$ for L simply connected and $\mathbb{Q}$-acyclic but not $\mathbb{Z}$-acyclic; then G is finitely presented and $\mathrm{FH}(\mathbb{Q})$, but not $\mathrm{FP}(\mathbb{Z})$.

Since $G \in \mathrm{FH}(\mathbb{Q})$, there is a finite complex K with $\pi_1 K = G$ and $\tilde{H}_\star(\tilde{K}; \mathbb{Q}) = 0$.

3.3.2 An acyclic reflection group trick

Mike Davis introduced his reflection group trick in [Dav83]; his excellent book [Dav08] is an great reference, and includes all the proofs of the facts we need here.

Since K is a finite complex, there is an embedding $K \hookrightarrow \mathbb{R}^N$ for some N; a *regular neighborhood* of $K \subset \mathbb{R}^N$ is manifold N with boundary ∂N. Observe that N deform retracts to K. An introduction to the theory of regular neighborhoods can be found in [RS72, Coh69].

The reflection group trick uses a Coxeter group to glue together copies of N, transforming the manifold with boundary N to a closed manifold W.

We now describe how the copies of N are glued together. Choose a flag triangulation L of ∂N; let G be the right-angled Coxeter group associated to this flag triangulation. For each vertex $v \in L$, let D_v be the star of v in the barycentric subdivision L' of L. Copies of N will be glued along "mirrors," namely the D_v; specifically, define

$$\tilde{W} = (N \times G)/\sim$$

where $(x, g) \sim (x, gh)$ whenever $x \in D_h$. Choose a finite index torsion-free subgroup G' of G, and let $W = \tilde{W}/G'$. An application of Mayer-Vietoris proves

Proposition 3.3.2. *W is a closed manifold, with $\mathbb{Q}$-acyclic universal cover.*

Additionally, $\pi_1 W = G' \rtimes \Gamma$ is torsion-free, since G' and Γ are both torsion-free (the former by assumption, the latter because it is an Artin group).

Proposition 3.3.3. *The Eilenberg-MacLane space $K(\pi_1 W, 1)$ does not have the homotopy type of a finite complex.*

Proof. The group $\pi_1 W = G' \rtimes \Gamma$ retracts onto Γ, and so, $K(\pi_1 W, 1)$ retracts onto $B\Gamma$. If $K(\pi_1 W, 1)$ had the homotopy type of a finite complex, then $B\Gamma$ would be a finitely dominated complex, and so $\Gamma \in \mathrm{FP}(\mathbb{Z})$. But Γ was constructed (using Bestvina–Brady Morse theory) so that $\Gamma \notin \mathrm{FP}(\mathbb{Z})$. $\square$

The existence of such a group W proves Theorem 3.1.1.

Having considered the torsion-free case, we now consider the situation with torsion.

CHAPTER 4
FINITENESS

4.1 Introduction

The first obstruction to a PD(R) group Γ being the fundamental group of an aspherical R-homology manifold is a finiteness obstruction: is $B\Gamma$ homotopy equivalent to a finite complex?

This is too restrictive a condition on a group Γ: for instance, if Γ has torsion, $B\Gamma$ never has the homotopy type of a finite complex. Nevertheless, groups with torsion can be PD(R) groups, although not when $R = \mathbb{Z}$.

To broaden the Poincaré duality groups available for study, we consider groups with torsion, but instead of hopelessly seeking a contractible space with a free action, we will look for an acyclic space with a free action of our given group containing torsion.

Question 4.1.1. For which groups Γ does there exist a finite complex X with

- $\bar{H}_\star(\tilde{X}; R) = 0$ and
- $\pi_1 X = \Gamma$?

In other words, which groups act "nicely" (e.g., properly discontinuously, cellularly, cocompactly) on acyclic complexes? This is property FH(R) of Bestvina and Brady, discussed earlier in section 2.1. Since FH(R) $\supset$ Mfld(R), satisfying FH(R) is neccessary for a PD(R) group to be a manifold group.

For ease of exposition, we will concentrate on virtually torsion-free groups, so there is a group extension

$$1 \to \pi \to \Gamma \to G \to 1.$$

The general case can be dealt with using additional "orbi-" terminilogy, but is not necessary for the examples of most interest to us (see Selberg's lemma below).

Assume $B\pi$ has the homotopy type of a finite complex (i.e., $\pi \in \mathrm{F}$), and G is a finite group, making $\Gamma \in \mathrm{VF}$. When does such a group act freely on an R-acyclic complex, that is, for which $\Gamma \in \mathrm{VF}$ is it the case that $\Gamma \in \mathrm{FH}(R)$? The finite group G already acts on the finite complex $B\pi$, but when can this action be improved to a free action on an acyclic space?

This amounts to a finiteness obstruction. W. Lück designed an equivariant finiteness theory [Lüc89]; in section 4.3 we will express his equivariant finiteness theory in more category theoretic language (similar to [DL98]), with the following result:

Theorem 4.1.2. *Let* $\Gamma \in \mathrm{VF}$*, so there is an extension*

$$1 \to \pi \to \Gamma \to G \to 1$$

with $B\pi$ *a finite complex and* G *a finite group;* G *acts on* $B\pi$ *(not fixing a base-point), and if, for all nontrivial subgroups* $H \subset G$*, and every connected component* C *of* $(B\pi)^H$*,*

$$\chi(C) = 0,$$

then $\Gamma \in \mathrm{FH}(\mathbb{Q})$*, i.e., there exists a compact space* X *with* $\pi_1 X = \Gamma$ *and* $\tilde{X}$ *rationally acyclic.*

Since crossing with S^1 guarantees $\chi = 0$, it is not too hard to produce examples: a group satisfying VF, after crossing with $\mathbb{Z}$, satisfies $\mathrm{FH}(\mathbb{Q})$.

In Section 4.2, we will also see that vanishing of certain Euler characteristics is necessary, namely $\chi((B\pi)^H)$ for cyclic subgroups H.

4.2 Lefschetz fixed point theorem

In order to take about Euler characteristics over various fields, we will write $\chi(X;R)$ to mean $\sum_i (-1)^i \dim H_i(X;R)$.

The Lefschetz fixed point theorem ([Hat02], [Bro71]) will obstruct some groups from satisfying FH(R).

Proposition 4.2.1. *Suppose R is a field, and that*

$$1 \to \pi \to \Gamma \to G \to 1,$$

with $B\pi$ homotopy equivalent to a finite complex, and G a finite group. If there exists a compact X having $\pi_1 X = \Gamma$ and R-acyclic $\tilde{X}$, then, for all nontrivial $g \in G$,

$$\chi\left((B\pi)^{\langle g \rangle}; R\right) = 0.$$

Proof. The map $\tilde{X}/\pi \to B\pi$ is an R-homology equivalence, and is G-equivariant (though not necessarily an equivariant homotopy equialvence). Consequently,

$$\begin{aligned}
\chi\left((B\pi)^{\langle g \rangle}; R\right) &= \operatorname{trace}\left(g_\star : H_\star(B\pi; R) \to H_\star(B\pi; R)\right) \quad \text{(by Lefschetz)} \\
&= \operatorname{trace}\left(g_\star : H_\star(\tilde{X}/\pi; R) \to H_\star(\tilde{X}/\pi; R)\right) \quad \text{(by G-equivariance)} \\
&= 0 \quad \text{(by freeness of the G-action on $\tilde{X}/\pi$)}.
\end{aligned}$$

□

Proposition 4.2.1 is strong enough to obstruct certain VF groups from satisfying FH($\mathbb{Q}$).

Example 4.2.2. The group $\mathbb{Z}/p\mathbb{Z}$ acts on $\mathbb{Z}^p$ by permuting coordinates; $\mathbb{Z}/p\mathbb{Z}$ also acts on the kernel of the map $\mathbb{Z}^p \to \mathbb{Z}$ given by adding coordinates. Use the action on the kernel to define $\Gamma = \mathbb{Z}^{p-1} \rtimes \mathbb{Z}/p\mathbb{Z}$. Since $\mathbb{Z}^{p-1} \in \mathrm{F}$, we have $\Gamma \in \mathrm{VF}$.

The action of $\mathbb{Z}/p\mathbb{Z}$ on $B\mathbb{Z}^{p-1} = (S^1)^{p-1}$ fixes p isolated points, so

$$\chi\left(\left((S^1)^{p-1}\right)^{\mathbb{Z}/p\mathbb{Z}}\right) = p,$$

and hence Proposition 4.2.1 implies that Γ does not act freely on any $\mathbb{Q}$-acyclic complex.

The proposition provides a necessary condition, but the given condition is far from sufficient. As we will see, a sufficient condition requires examining the Euler chacteristic of connected components of other subgroups, not just the cyclic subgroups.

4.3 Finiteness over a category

4.3.1 Categories over categories

Definition 4.3.1. Let $\mathcal{C}$ be a small category, and **Cat** any category; we define a category $\mathcal{C}$-**Cat**.

- $\operatorname{Obj} \mathcal{C}$-**Cat** consists of functors $F : \mathcal{C} \to \mathbf{Cat}$, and
- given two such functors F and G, the morphisms $\operatorname{Hom}_{\mathcal{C}\text{-}\mathbf{Cat}}(F, G)$ are the natural transformations from F to G.

This is the *functor category* and is usually denoted $\mathbf{Cat}^{\mathcal{C}}$; we use the alternate notation $\mathcal{C}$-**Cat**, analogous to the equivariant notation for G-spaces.

Example 4.3.2. The category **Spaces** is the category of compactly generated topological spaces; an object in $\mathcal{C}$-**Spaces** is called a (covariant) $\mathcal{C}$-space; a contravariant $\mathcal{C}$-space is just a covariant $\mathcal{C}^{\mathrm{op}}$-space. We likewise have $\mathcal{C}$-**AbGroups** and $\mathcal{C}$-R-**Mod**, which form abelian categories and for which W. Lück has developed homological algebra [Lüc89].

Proposition 4.3.3. *Any functor* $F : \boldsymbol{A} \to \boldsymbol{B}$ *induces*

$$\mathcal{C}\text{-}F : \mathcal{C}\text{-}\boldsymbol{A} \to \mathcal{C}\text{-}\boldsymbol{B}$$

by sending $A : \mathcal{C} \to \boldsymbol{A}$ *to* $\mathcal{C}\text{-}F(A) = F \circ A$.

For instance, the fundamental groupoid functor Π : **Spaces** $\to$ **Groupoids** induces

$$\mathcal{C}\text{-}\Pi : \mathcal{C}\text{-}\mathbf{Spaces} \to \mathcal{C}\text{-}\mathbf{Groupoids}.$$

At times, however, we would like to talk about the category of $\mathcal{C}$-spaces, for a varying small category $\mathcal{C}$. This desire is behind the following definition.

Definition 4.3.4. Let $\mathbf{A}$ and $\mathbf{B}$ be categories, with $\mathbf{A}$ a subcategory of $\mathbf{Cat}$, the category of small categories. Then the category

$$\mathbf{A} \downarrow \mathbf{B}$$

has as objects the functors $F : A \to \mathbf{B}$, for A an object of $\mathbf{A}$; in other words, an object of the category $\mathbf{A} \downarrow \mathbf{B}$ consists of a choice of an object $A \in \mathbf{A}$, and a functor from A to $\mathbf{B}$.

The morphisms $\mathrm{Hom}_{\mathbf{A}\downarrow\mathbf{B}}(F : A \to \mathbf{B}, F' : A' \to \mathbf{B})$ consist of a functor $H : A \to A'$ with a natural transformation from F to $F' \circ H$.

Remark 4.3.5. Although it will not be important in the sequel, note that $\mathbf{A}$ is a 2-category (in the sense of a category enriched over $\mathbf{Cat}$), and $\mathbf{A} \downarrow \mathbf{B}$ is likewise a 2-category.

Example 4.3.6. Consider the subcategory of $\mathbf{Cat}$ containing a single small category $\mathcal{C}$ and the identity functor; by abuse of notation, we call also this category $\mathcal{C}$. Then $\mathcal{C} \downarrow \mathbf{B}$ is the same thing as $\mathcal{C}$-$\mathbf{B}$ constructed in Definition 4.3.1

4.3.2 Balanced products

A construction well-known to category theorists—that of a *coend*—gives a natural transformation from a bifunctor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Cat}$ to a constant functor [Mac71]. We apply this in the case of $\mathbf{Cat}$, a monoidal category, to combine a contravariant and covariant $\mathcal{C}$-object over $\mathbf{Cat}$ into an object of $\mathbf{Cat}$.

Definition 4.3.7. Let $\mathbf{Cat}$ be a monoidal category with product $\times$; let A and B be contravariant and covariant $\mathcal{C}$-objects, respectively. Then the *balanced product* of A and B, written $A \times_{\mathcal{C}} B$, is

$$\coprod_{c \in \mathrm{Obj}\,\mathcal{C}} A(c) \times B(c) / \sim$$

where $(xf, y) \sim (x, fy)$ for $x \in A(d)$, $y \in B(c)$, and $f \in \mathrm{Hom}(c, d)$.

We will be using balanced product in the context of spaces (under cartesian product of spaces) and modules (under tensor product of modules).

The balanced product seems a bit abstract, but the balanced product (and coends more generally) is a abstraction of a well-known construction: geometric realization.

Example 4.3.8. Define $\mathbf{\Delta}$, the simplicial category (see [May67]), where

- $\mathrm{Obj}\,\mathbf{\Delta}$ consists of totally ordered finite sets, and
- $\mathrm{Hom}_{\mathbf{\Delta}}(A, B)$ consists of order-preserving functions from A to B.

Further define Δ to be the $\mathbf{\Delta}$-space, sending a totally ordered finite A to

$$\Delta(A) = (|A| - 1)\text{-simplex},$$

and an order-preserving function to the inclusion of simplices.

A *simplicial space* is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Spaces}$, i.e., an object of $\mathbf{\Delta}^{\mathrm{op}}$-$\mathbf{Spaces}$. The balanced product of a simplicial space X with Δ (written $X \times_{\mathbf{\Delta}} \Delta$), is the *geometric realization* of the simplicial space X.

Example 4.3.9. Another instance of this construction appears when handling a *complex of groups* (an introduction to which appears in [Dav02]). That is, given a simplicial group, i.e., a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Groups}$, we want to put these groups together to form a single group—i.e., we consider $X \times_{\mathbf{\Delta}} \Delta$, which is a group.

4.3.3 Orbit category

Definition 4.3.10. Define the *orbit category* of a group G, written $\mathrm{Or}(G)$, as follows:

- $\mathrm{Obj}\,\mathrm{Or}(G) = \{G/H : H \text{ a subgroup of } G\}$,
- $\mathrm{Hom}_{\mathrm{Or}(G)}(G/H, G/K)$ is the set of G-maps between the G-sets G/H and G/K.

Naturally associated to a G-space, there are both a contravariant and covariant $\mathrm{Or}(G)$-spaces.

Example 4.3.11. Let X be a (left) G-space; there is a contravariant $\mathrm{Or}(G)$-space

$$G/H \mapsto X^H,$$

with $G/H \to G/H'$ sent to $X^{H'} \subset X^H$.

Associated to X, there is also a covariant $\mathrm{Or}(G)$-space

$$G/H \mapsto X/H,$$

with $G/H \to G/H'$ sent to $X/H \to X/H'$.

In fact, the reverse is possible: given a a contravariant $\mathrm{Or}(G)$-space, we can recover a G-space.

Proposition 4.3.12. *A contravariant* $\mathrm{Or}(G)$*-space is (naturally) a left* G*-space.*

Proof. The construction is formally similar to geometric realization (see Example 4.3.8).

Suppose X is a contravariant $\mathrm{Or}(G)$-space. Let ∇ be the covariant $\mathrm{Or}(G)$-space given by sending G/H to itself, that is, to the finite set with the discrete topology. Then

$$X \otimes_{\mathrm{Or}(G)} \nabla$$

is a (left) G-space. Specifically, $g \in G$ acts on $X \otimes_{\mathrm{Or}(G)} \nabla$ by the map $\mathrm{id} \otimes_{\mathrm{Or}(G)} L_g$ where $L_g : G/H \to G/H$ is left multiplication by g. □

4.3.4 K-theory

An object in **Groupoids** $\downarrow$ R-**Mod** is an "$R[G]$-module" for some groupoid G; we define certain (full) subcategories of **Groupoids** $\downarrow$ R-**Mod**, corresponding to finitely generated free and finitely generated projective $R[G]$-modules. We will speak of both contravariant and covariant $R[G]$-modules.

Definition 4.3.13. A *complex* in **Groupoids** $\downarrow$ R-**Mod** is a collection of such modules M_i (with $i \in \mathbb{Z}$) and maps $d_i : M_i \to M_{i-1}$. We say the complex is *bounded* if all but finitely many of the modules are zero.

Write **Cplx (Groupoids** $\downarrow$ R-**Mod)** for the category of complexes of finitely generated projective R-modules over a groupoid; maps between complexes are *chain maps*.

As is usually the case, "free" is adjoint to "forgetful" (i.e., the forgetful functor from **Groupoids** $\downarrow$ R-**Mod** to **Groupoids** $\downarrow$ **Sets**).

Definition 4.3.14 (see page 167, [Lüc89]). A module M in **Groupoids** $\downarrow$ R-**Mod** is a *free module* with basis $B \subset M$, an object in **Groupoids** $\downarrow$ **Sets**, if, for any object N in **Groupoids** $\downarrow$ R-**Mod** and map $f : B \to N$, there is a unique morphism $F : M \to N$ extending f.

In addition to free modules with basis B, we can speak about modules generated by a particular subset.

Definition 4.3.15 (see page 168, [Lüc89]). Suppose M is an object in the category **Groupoids** $\downarrow$ R-**Mod**, and S is a subset (i.e., an object in **Groupoids** $\downarrow$ **Sets**). Then the *span* of S is the smallest module containing S, namely,

$$\operatorname{span} S = \bigcap \{N : S \subset N \text{ and } N \text{ is a submodule of } M\}.$$

If S is a finite set (i.e., finite over the indexing category, meaning $S(g)$ is a finite set for each object g in the groupoid), we say that $\operatorname{span} S$ is *finitely generated*.

Definition 4.3.16 (see page 169, [Lüc89]). A module P in **Groupoids** $\downarrow$ R-**Mod** is *projective* if either of the following equivalent conditions holds:

- Each exact sequence $0 \to M \to N \to P \to 0$ splits.
- P is a direct summand of a free module.

Having studied these modules, we can define an appropriate K-theory for the category **Groupoids** $\downarrow$ R-**Mod**, via Waldhausen categories [Wal85] as in [Lüc89]. This K-theory is the correct receiver for the Euler characteristic.

Definition 4.3.17. The *Euler characteristic* χ of a bounded complex (M_i, d_i) in **Cplx** (**Groupoids** $\downarrow$ R-**Mod**) is

$$\chi(\cdots \to M_0 \to) = \sum_{i\in\mathbb{Z}} (-1)^i [M_i] \in K_0(\mathbf{Groupoids} \downarrow R\textbf{-Mod}).$$

4.3.5 Chain complex of the universal cover

In Wall's finiteness obstruction for a space X, the most important object is $\tilde{C}(X)$, the $R[\pi_1 X]$-chain complex of the universal cover of X. This is traditionally denoted by $C_\star(\tilde{X}; R)$, but we will write $\tilde{C}(X)$ to emphasize the functorial nature of the construction.

However, the usual construction is insufficiently functorial: $\tilde{C}$ transforms a space X into a chain complex over a ring that depends on the group $\pi_1 X$; consequently, it is not clear what the target category of $\tilde{C}$ ought to be. Worse, only basepoint preserving maps $X \to X$ induce endomorphisms of $\tilde{C}(X)$.

The definition of $\mathbf{A} \downarrow \mathbf{B}$ is exactly what we need to define the target of the functor $\tilde{C}$, and by using the fundamental groupoid instead of the fundamental group, we avoid the basepoint issue: *any* self-map of X will induce a self-map of $\tilde{C}(X)$.

Before we can define $\tilde{C}$, we define the *universal cover functor*. The functor $\tilde{-}$: **Spaces** $\to$ **Groupoids** $\downarrow$ **Spaces** sends a space X to the functor $\tilde{X} : \Pi X \to$ **Spaces**. This latter functor sends a object in ΠX, which is just a point $x \in X$, to the universal cover of X using x as the base point.

The functor C : **Spaces** $\to$ **Cplx** (R-**Mod**) sends a space to its singular R-chain complex. Note that this induces a functor

$$\mathbf{Groupoids} \downarrow C : \mathbf{Groupoids} \downarrow \mathbf{Spaces} \to \mathbf{Groupoids} \downarrow \mathbf{Cplx}\,(R\textbf{-Mod})$$

Not too surprisingly, we compose $\tilde{-}$ and **Groupoids** $\downarrow C$.

Definition 4.3.18 (See page 259, [Lüc89]). The functor

$$\tilde{C} : \mathbf{Spaces} \to \mathbf{Cplx}\,(\mathbf{Groupoids} \downarrow R\text{-}\mathbf{Mod})$$

sends a space X to $C(\tilde{X})$. In other words, $\tilde{C}$ is the composition of functors

$$\mathbf{Spaces} \xrightarrow{\tilde{-}} \mathbf{Groupoids} \downarrow \mathbf{Spaces} \xrightarrow{\mathbf{Groupoids}\downarrow C} \mathbf{Groupoids} \downarrow \mathbf{Cplx}\,(R\text{-}\mathbf{Mod})\,.$$

Note that there is a natural map

$$\mathbf{Groupoids} \downarrow \mathbf{Cplx}\,(R\text{-}\mathbf{Mod}) \to \mathbf{Cplx}\,(\mathbf{Groupoids} \downarrow R\text{-}\mathbf{Mod})\,.$$

As a result of the improved functoriality of $\tilde{C}$, we can apply $\tilde{C}$ over a small category $\mathcal{C}$ (via Proposition 4.3.3), to get

$$\begin{aligned} \mathcal{C}\text{-}\tilde{C} : \mathcal{C}\text{-}\mathbf{Spaces} &\to \mathcal{C}\text{-}\mathbf{Cplx}\,(\mathbf{Groupoids} \downarrow R\text{-}\mathbf{Mod}) \\ &\to \mathbf{Cplx}\,(\mathcal{C} \downarrow (\mathbf{Groupoids} \downarrow R\text{-}\mathbf{Mod}))\,. \end{aligned}$$

4.3.6 Instant finiteness obstruction

Our goal is to define maps

$$\begin{aligned} \mathrm{Wall} : \mathbf{FindomSpaces} &\to K_0(\mathbf{Groupoids} \downarrow R\text{-}\mathbf{Mod}), \\ \widetilde{\mathrm{Wall}} : \mathbf{FindomSpaces} &\to \tilde{K}_0(\mathbf{Groupoids} \downarrow R\text{-}\mathbf{Mod}), \end{aligned}$$

so that $\widetilde{\mathrm{Wall}} \neq 0$ obstructs an R-finitely dominated space from being R-homotopy equivalent to a finite complex. There are a few terms that need to be defined.

Here, **FindomSpaces** is a built from a full subcategory of **Spaces**, consisting of those spaces which are *R-finitely dominated*, but we the choice of domination is part of the data.

Definition 4.3.19. A space Y is *R-dominated* by X if there are maps

$$Y \xrightarrow{i} X \xrightarrow{r} Y$$

with $r \circ i : Y \to Y$ an R-homotopy equivalence.

Definition 4.3.20. A space is *R-finitely dominated* by X if X is a finite complex.

Whenever we speak of an R-homotopy equivalence, we really mean an $R[\pi_1]$ equivalence—i.e., the induced map $\tilde{C}(Y) \to \tilde{C}(Y)$ is chain homotopic to the identity.

Ranicki defined an *instant finiteness obstruction* [Ran85]. Importantly, his algebraic framework remains applicable even for complexes of modules over a category. We will use this "instant" perspective to define the maps Wall and $\widetilde{\text{Wall}}$ for a finitely dominated space. Say that Y is R-finitely dominated by X with $Y \xrightarrow{i} X \xrightarrow{r} Y$. Then the map $i \circ r : X \to X$ induces $\tilde{C}(i \circ r) : \tilde{C}(X) \to \tilde{C}(X)$, the image of which is a chain complex of projective modules. We define $\text{Wall}(Y)$ to be $\chi(\tilde{C}(i \circ r))$.

Note that $\widetilde{\text{Wall}}(Y)$ does not depend on the choice of domination, but $\text{Wall}(Y)$ definitely does, and should be thought of as a sort of Euler characteristic.

The machinery developed in [Ran85] can be adapted to prove

Proposition 4.3.21. *If a space X is R-finitely dominated, and $\widetilde{\text{Wall}}(X) = 0$, then X is R-homotopy equivalent to a finite complex.*

Or rather, we can show that $\tilde{C}(X)$ is chain equivalent to complex of finitely generated free R-modules. In many cases, this is enough: Leary (in Theorem 9.4, [Lea02]), shows that if G is a group of finite type (i.e., BG has finitely many cells in each dimension), then G is $\text{FL}(\mathbb{Q})$ if and only if G is $\text{FH}(\mathbb{Q})$. This means that finite domination and the above algebra suffices to get the geometry.

What we have done thus far for spaces is valid for $\mathcal{C}$-spaces. For instance, a finitely dominated $\mathcal{C}$-space is dominated by a finite $\mathcal{C}$-space X, meaning that for each $c \in \mathcal{C}$, the space $X(c)$ is finite.

Proposition 4.3.22. *If Y and Y' are R-finitely dominated cotra- and covariant (respectively) $\mathcal{C}$-spaces and $\mathcal{C}$ is finite, then*

$$Y \times_{\mathcal{C}} Y'$$

is an R-finitely dominated space.

Proof. This is fairly straightforward: suppose Y and Y' are finitely dominated by X and X', respectively. Then we have

$$Y \times_{\mathcal{C}} Y' \xrightarrow{i \times_{\mathcal{C}} i'} X \times_{\mathcal{C}} X' \xrightarrow{r \times_{\mathcal{C}} r'} Y \times_{\mathcal{C}} Y'$$

and it is enough to prove that

$$(r \times_{\mathcal{C}} r') \circ (i \times_{\mathcal{C}} i') : \tilde{C}(Y \times_{\mathcal{C}} Y') \to \tilde{C}(Y \times_{\mathcal{C}} Y')$$

is an R-equivalence, and that $X \times_{\mathcal{C}} X'$ is a finite complex. □

The finiteness obstruction for a balanced product $Y \times_{\mathcal{C}} Y'$ can be computed from the finiteness obstructions of the terms Y and Y'; this amounts to an equivariant version of the Eilenberg–Zilber theorem, as in [GG99].

Proposition 4.3.23. *For Y and Y', finitely dominated contravariant and covariant $\mathcal{C}$-spaces, respectively,*

$$\mathrm{Wall}(Y \times_{\mathcal{C}} Y') = \mathrm{Wall}(Y) \otimes_{\mathcal{C}} \mathrm{Wall}(Y').$$

This follows from calculations in [Lüc89] (for example, see page 229) relating balanced tensor product of chain complexes to the balanced product of spaces.

4.4 Equivariant finiteness

Let X be a G-space; we consider X to be a contravariant $\mathrm{Or}(G)$-space by Proposition 4.3.12.

Definition 4.4.1. $B\,\mathrm{Or}(G)$ is the covariant $\mathrm{Or}(G)$-space given by

$$G/H \mapsto BH = K(H,1),$$

and sending the map $G/H \to G/H'$ to the map $BH \to BH'$ induced from $H \subset H'$.

Proposition 4.4.2. *For a G-space X,*

$$X \times_{\mathrm{Or}(G)} B\,\mathrm{Or}(G)$$

is associated to the G-space $X \times_G BG = (X \times EG)/G$.

Proposition 4.4.3. *The finiteness obstruction* $\mathrm{Wall}(X \times_{\mathrm{Or}(G)} B\,\mathrm{Or}(G))$ *vanishes provided*

$$\chi(\textit{connected component of } X^H) = 0$$

for all nontrivial subgroups $H \subset G$.

Proof. To calculate

$$\mathrm{Wall}(X \times_{\mathrm{Or}(G)} B\,\mathrm{Or}(G)) = \mathrm{Wall}(X) \otimes_{\mathrm{Or}(G)} \mathrm{Wall}(B\,\mathrm{Or}(G))$$

note that X is already finite, so $\mathrm{Wall}(X)$ is an equivariant Euler characteristic. Additionally, since each H is finite, the rational chain complex of BH can be taken to be the single module $[\mathbb{Q}]$ in degree 0, since $[\mathbb{Q}]$ is projective as a $\mathbb{Q}H$ module.

In other words, $\mathrm{Wall}(B\,\mathrm{Or}(G))(G/H)$ is $[\mathbb{Q}]$. If all the Euler characteristics of all components of fixed sets of X vanish, then the balanced tensor product also vanishes—the vanishing Euler characteristic kills each $[\mathbb{Q}]$. □

4.5 Applications

There are examples where the equivariant finiteness obstruction vanishes; in this section, we will give two sources of such examples: the reflection group trick, and lattices with torsion.

4.5.1 Reflection group trick

For a description of the reflection group trick, see Section 3.3.2 or [Dav83, Dav08]. To illustrate the trick, we construct a group Γ, with n-torsion, which is the fundamental group of a rational homology manifold (in fact, a manifold) with $\mathbb{Q}$-acyclic universal cover. Since the fundamental group Γ has n-torsion, there is no closed, aspherical manifold with fundamental group Γ. Succinctly, we construct $\Gamma \in \mathrm{Mfld}(\mathbb{Q})$ with $\Gamma \notin \mathrm{AsphMfld}(\mathbb{Q})$, and therefore, also not in $\mathrm{AsphMfld}(\mathbb{Z})$.

Proposition 4.5.1. *Let $G = \mathbb{Z} \times \mathbb{Z}_n$. Then there exists a closd manifold M so that*

- *$\pi_1 M$ retracts onto G,*
- *$\tilde{M}$ is $\mathbb{Q}$-acyclic.*

Proof. Let $\pi = \mathbb{Z}$, so that $B\mathbb{Z} = S^1$. Consider $B\mathbb{Z}$ with the trivial $\mathbb{Z}_n$ action, so that the fixed set $(B\mathbb{Z})^{\mathbb{Z}_n} = B\mathbb{Z} = S^1$, and therefore has vanishing Euler characteristic. By the equivariant finiteness theory, there is a finite complex Y having $\pi_1 Y = \mathbb{Z} \times \mathbb{Z}_n$ and having universal cover $\tilde{Y}$ rationally acyclic. In other words, $\mathbb{Z} \times \mathbb{Z}_n \in \mathrm{FH}(\mathbb{Q})$.

Of course, for this basic example we do not have to apply the general theory: it is not that hard to construct, by hand, such a space Y.

Since Y is finite, it embeds in $\mathbb{R}^N$ for some N, and we can apply the reflection group trick to a regular neighborhood of $Y \subset \mathbb{R}^N$ (see Section 3.3.2), producing a manifold (not just a rational homology manifold!) M with universal cover $\tilde{M}$ rationally acyclic, and $\pi_1 M$ retracting onto G. □

Since $\pi_1 M$ retracts onto a group containing n-torsion, $\pi_1 M$ also contains n-torsion.

Corollary 4.5.2. *There are groups satisfying* $\mathrm{Mfld}(\mathbb{Q})$ *which contain n-torsion for any n.*

In fact, the same proof works for any finite group—not just $\mathbb{Z}_n$.

Corollary 4.5.3. *For every finite group G, there exists $\Gamma \in \mathrm{Mfld}(\mathbb{Q})$ with $G \subset \Gamma$.*

4.5.2 Preliminaries on Lattices

Historically, the first source of Poincaré duality groups were fundamental groups of aspherical manifolds, and a basic source of aspherical manifolds are lattices.

Proposition 4.5.4. *Let G be a semisimple Lie group, K a maximal compact, and Γ a torsion-free uniform lattice (i.e., a discrete cocompact subgroup). Then $\Gamma \backslash G/K$ is a compact, aspherical manifold with fundamental group Γ.*

For an introduction to locally symmetric spaces such as $\Gamma \backslash G/K$, see [Hel62].

In other words, uniform torsion-free lattices Γ satisfy $\mathrm{Mfld}(\mathbb{Z})$, and therefore, satisfy $\mathrm{PD}(\mathbb{Z})$. Next, we examine what happens when Γ satisfies the weaker condition $\mathrm{VPD}(\mathbb{Z})$.

Proposition 4.5.5. *Let G be a finite group, π a group satisfying $\mathrm{PD}(\mathbb{Z})$, and Γ an extension,*

$$1 \to \pi \to \Gamma \to G \to 1.$$

Then Γ satisfies $\mathrm{PD}(\mathbb{Q})$.

One can do better than $\mathbb{Q}$: if $R = \mathbb{Z}[1/G]$, meaning $\mathbb{Z}$ with divisors of $|G|$ inverted, then Γ is $\mathrm{PD}(R)$.

Proof. Extensions of Poincaré duality groups by Poincaré duality groups satisfy Poincaré duality [JW72], and finite groups are 0-dimensional $\mathbb{Q}$-Poincaré duality groups. □

Understanding groups satisfying $\mathrm{VPD}(\mathbb{Z})$ permits us to examine linear groups with torsion, by applying Selberg's lemma [Sel60].

Lemma 4.5.6 (Selberg). *Every finitely generated linear group contains a finite index normal torsion-free subgroup (in other words, is virtually torsion-free).*

Example 4.5.7. Uniform lattices, even when they contain torsion, satisfy PD($\mathbb{Q}$) because, by Selberg's lemma, a uniform lattice is virtually torsion-free, and therefore, satisfies VPD($\mathbb{Z}$).

Whether $\chi(\Gamma\backslash G/K)$ vanishes is indepedent of Γ; it depends only on the Lie group G. This is true even if Γ is non-uniform (via measure equivalence [Gab02] and the equality of the L^2 and usual Euler characteristic [Ati76, Lüc02]). The fixed sets are themselves lattices in smaller Lie groups, so it is easy to check that the Euler characteristic vanishes on fixed sets. As a result, lattices form a particularly nice class with respect to the finiteness obstructions from Section 4.4.

4.5.3 Vanishing obstructions

It is not difficult to build explicit examples of groups for which the obstructions vanish.

Proposition 4.5.8. *There is a uniform arithmetic lattice π in* $\mathrm{SO}(p,1)$ *and a* $\mathbb{Z}_p$ *action on the locally symmetric space*

$$X = \pi\backslash \mathrm{SO}(p,1)/\mathrm{SO}(p)$$

with fixed set $X^{\mathbb{Z}_p} = S^1$.

Proof. We first recall the usual construction of arithmetic lattices; we follow Chapter 15C of [Mor01] and describe how to produce an arithmetic lattice in $\mathrm{SO}(p,1)$. Begin by defining a bilinear form

$$B(x,y) = \sum_{i=1}^{p} x_i\, y_i - \sqrt{2}\, x_0\, y_0$$

so that $G = \mathrm{SO}(B) = \mathrm{SO}(p,1)$. Note that $\mathbb{Z}_p$ acts on $\mathbb{R}^{p+1}$ preserving this form, and that the action is by integer matrices. Define the lattice $\pi = G_{\mathcal{O}}$ for $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$.

The diagonal map $\Delta : G \to G \times G^\sigma$, for σ the Galois automorphism of $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$, sends π to $\Delta(\pi)$, a lattice in $G \times G^\sigma$. But $G^\sigma = \mathrm{SO}(p+1)$ is compact, so after quotienting, π is still a lattice in G. And π is cocompact, by the Godement Compactness Criterion (that arithmetic lattices are cocompact precisely when they have no nontrivial unipotents [MT62]).

The action of $\mathbb{Z}_p$ descends to the quotient (as it preserves the lattice). In the universal cover $\mathrm{SO}(p,1)/\mathrm{SO}(p)$, the set fixed by $\mathbb{Z}_p$ is a line; in the quotient X, it is possible that the action might fix additional points—but it does not, as the set fixed by $\mathbb{Z}_p$ is no more than 1 dimensional, and $\mathbb{Z}_p$ cannot fix isolated points on an odd-dimensional manifold. So the fixed set is a 1-manifold, i.e., a disjoint union of circles. □

By Proposition 4.5.8, there is an extension

$$1 \to \pi \to \Gamma \to \mathbb{Z}_p \to 1$$

and since $\chi(B\pi^{\mathbb{Z}_p}) = \chi(S^1) = 0$, the equivariant finiteness theory implies that there exists a space Y with $\pi_1 Y = \Gamma$ and whose the universal cover $\tilde{Y}$ is a rationally acyclic space. In short, $\Gamma \in \mathrm{FH}(\mathbb{Q})$.

Question 4.5.9. For which n does $\mathbb{Z}/p\mathbb{Z}$ act with nontrivial fixed set on an hyperbolic n-manifold?

This is possible in dimensions 2 and 3 by taking branched covers (as in [GT87]). Asking for a nontrivial fixed set is important: Belolipetsky and Lubotzky [BL05] have shown that for $n \geq 2$, every finite group acts *freely* on a compact hyperbolic n-manifold.

In contrast, the construction in Proposition 4.5.8 required $n \geq p$ to get $\mathbb{Z}_p$ to act with nontrivial fixed set. The fact that there are only finitely many arithmetic triangle groups [Tak77] is perhaps relevant to answering this question.

If we relax Question 4.5.9 to a combinatorial curvature condition (i.e., locally $\mathrm{CAT}(-1)$; see Section 3.2.3), we can easily prove the following.

Proposition 4.5.10. *For every p and odd $n \geq 3$, there is a locally* CAT(-1) *manifold M admitting a $\mathbb{Z}_p$ action having fixed set $M^{\mathbb{Z}_p}$ a disjoint union of circles.*

Proof. In brief, first construct an action of $\mathbb{Z}_p$ on a closed n-manifold X^n, having fixed set a disjoint union of circles, and finish by hyperbolizing. Now we spell out a few details.

Our construction of X^n depends on our assumption that n is odd; in this case, $\mathbb{Z}_p$ acts freely on the odd-dimensional sphere S^{n-2}, and by taking the join with a circle on which $\mathbb{Z}_p$ acts trivially, we get an action of $\mathbb{Z}_p$ on $S^n = S^{n-2} * S^1$ having fixed set S^1.

Triangulate X equivariantly; consequently, the fixed set S^1 is in the 1-skeleton of X.

Now apply strict hyperbolization [CD95, DJ91] to the triangulation of X. The hyperbolized space inherits a $\mathbb{Z}_p$ action (since hyperbolization is functorial with respect to injective simplicial maps). The 1-skeleton of the hyperbolization of X consists of two copies of $X^{(1)}$, so a fixed circle in X contributes two circles to the hyperbolization. □

We cannot do something similar for even dimensional manifolds (because a $\mathbb{Z}_p$ action with circle fixed set would give, by considering the link of a fixed point, a $\mathbb{Z}_p$ action on an odd dimensional sphere with two fixed points, which is not by possible). But by crossing the output of Proposition 4.5.10 with S^1 on which $\mathbb{Z}_p$ acts trivially, we produce an even dimensional CAT(0) manifold with a $\mathbb{Z}_p$ action fixing a disjoint union of tori. That is, we have shown

Corollary 4.5.11. *For every p and every $n \geq 3$, there is a locally* CAT(0) *manifold M admitting a $\mathbb{Z}_p$ action having non-empty fixed set with vanishing Euler characteristic.*

4.5.4 Shrinking the family

It would, of course, be beneficial if, in calculating the finiteness obstruction for Γ,

$$1 \to \pi \to \Gamma \to G \to 1,$$

we did not have to consider *all* the nontrivial subgroups of G. Indeed, the following is true:

Proposition 4.5.12. *The finiteness obstruction* $\mathrm{Wall}(X \times_{\mathrm{Or}(G)} B\,\mathrm{Or}(G))$ *is a finite order element in* K_0*, provided*

$$\chi(\textit{connected component of } X^H) = 0$$

for all cyclic subgroups $H \subset G$.

To see this, we need only prove 4.4.3 somewhat more carefully—noting that the cyclic subgroups are minimal. Alternatively, this could be seen by making use of the splittings proved in [Lüc89]; Lück splits the equivariant K-theory, and proves that the splitting commutes with the finiteness obstruction. Rationally, this splitting can be seen to depend only on the cyclic subgroups.

A natural question, then, is whether all finite order elements of $K_0(\mathbb{Q}\Gamma)$ are, in fact, trivial. As we show in the following section, this is not the case.

4.5.5 Torsion in K_0

Conjecturally, for torsion-free groups Γ, the reduced K-theory $\tilde{K}_0(\mathbb{Q}\Gamma)$ vanishes; in other words, every projective $\mathbb{Q}\Gamma$-module is stably free. For groups containing torsion, this is not the case: I. Leary proved that for every integer n, and every field k of characteristic zero, there is a (virtually free) group G for which $\tilde{K}_0(kG)$ contains n-torsion [Lea02].

Here we explain an older example, originally due to Kropholler and Moselle [KM91]. Consider $(\mathbb{Z}_2)^2$ acting on $\mathbb{Z}^3$, given by

$$a \cdot (x, y, z) = (-x, -y, z),$$
$$b \cdot (x, y, z) = (x, -y, -z).$$

where a and b generate $(\mathbb{Z}_2)^2$. Let Γ be the extension

$$1 \to \mathbb{Z}^3 \to \Gamma \to (\mathbb{Z}_2)^2 \to 1.$$

We now compute $K_0(\mathbb{Q}\Gamma)$; in particular, note that $K_0(\mathbb{Q}\Gamma)$ contains 2-torsion.

Lemma 4.5.13. $K_0(\mathbb{Q}\Gamma) = \mathbb{Z}^{24} \oplus \mathbb{Z}_2$.

Proof. The Farrell–Jones conjecture holds for Γ with $\mathbb{Q}$-coefficients, and the edge homomorphism in the equivariant version of the Atiyah–Hirzebruch spectral sequence (see [KL05]) gives

$$\operatorname*{colim}_{\Gamma/H \in \mathrm{Or}(\Gamma;\mathcal{F}in)} \mathbb{K}_0(\mathbb{Q}H) \cong K_0(\mathbb{Q}\Gamma).$$

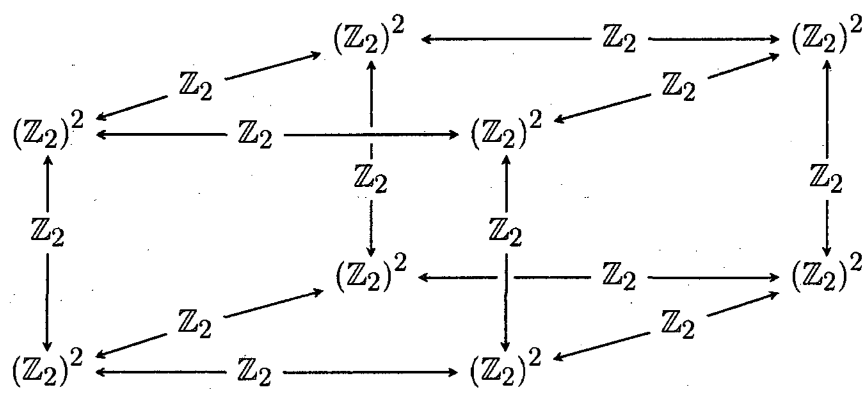
The finite subgroups of Γ are either trivial, $\mathbb{Z}_2$, or $(\mathbb{Z}_2)^2$. The group Γ contains

- twelve conjugacy classes of subgroups isomorphic to $(\mathbb{Z}_2)$, and
- eight conjugacy classes of subgroups isomorphic to $(\mathbb{Z}_2)^2$.

Figure 4.1 illustrates the inclusions between the conjugacy classes of finite subgroups. That a cube appears should not be surprising: Γ is an index two subgroup of $(D_\infty)^3$, and a model for $E_{\mathcal{F}in}(D_\infty)^3$ is $T^3 = (S^1)^3$, so $E_{\mathcal{F}in}(D_\infty)^3/(D_\infty)^3$, as an orbifold, is a cube with mirror faces.

Next we calculate the K-theory for the finite subgroups of Γ. Note that $K_0(\mathbb{Q}[\mathbb{Z}_2]) = \mathbb{Z}^2$ is generated by projective $\mathbb{Q}[\mathbb{Z}_2]$-modules $\mathbb{Q}_+$ and $\mathbb{Q}_-$. These modules are isomorphic to $\mathbb{Q}$ as $\mathbb{Q}$-modules; in the former, $\mathbb{Q}_+$, the generator of

Figure 4.1: Conjugacy classes of finite subgroups of Γ



$\mathbb{Z}_2$ acts trivially, and in the latter, $\mathbb{Q}_-$, the generator of $\mathbb{Z}_2$ acts by multiplication by -1. Similarly, $K_0(\mathbb{Q}[(\mathbb{Z}_2)^2]) = \mathbb{Z}^4$ is generated by

$$\mathbb{Q}_{++}, \quad \mathbb{Q}_{+-}, \quad \mathbb{Q}_{-+}, \quad \mathbb{Q}_{--},$$

where the subscripts record how the two generators of $(\mathbb{Z}_2)^2$ act on $\mathbb{Q}$, namely either trivially or by multiplication by -1.

Since the 8 subgroups $(\mathbb{Z}_2)^2$ are maximal among finite subgroups, the colimit will be a quotient of 8 copies of $K_0(\mathbb{Q}[(\mathbb{Z}_2)^2])$; in other words, a class in $K_0(\mathbb{Q}\Gamma)$ is given by classes in $K_0(\mathbb{Q}[(\mathbb{Z}_2)^2])$, up to equivalences given by the inclusions of $K_0(\mathbb{Q}[\mathbb{Z}_2])$ and $K_0(\mathbb{Q}[e])$. So we must study these inclusions.

The maps $K_0(\mathbb{Q}[\mathbb{Z}_2]) \to K_0(\mathbb{Q}[(\mathbb{Z}_2)^2])$ depend on which inclusion of $\mathbb{Z}_2$ into

$(\mathbb{Z}_2)^2$ we are considering:

$$\begin{aligned}
&\mathbb{Q}_+ \mapsto \mathbb{Q}_{++} \oplus \mathbb{Q}_{-+} \text{ for edges parallel to the } x\text{-axis} \\
&\mathbb{Q}_- \mapsto \mathbb{Q}_{--} \oplus \mathbb{Q}_{+-} \\
&\mathbb{Q}_+ \mapsto \mathbb{Q}_{++} \oplus \mathbb{Q}_{+-} \text{ for edges parallel to the } y\text{-axis} \\
&\mathbb{Q}_- \mapsto \mathbb{Q}_{-+} \oplus \mathbb{Q}_{--} \\
&\mathbb{Q}_+ \mapsto \mathbb{Q}_{++} \oplus \mathbb{Q}_{--} \text{ for edges parallel to the } z\text{-axis} \\
&\mathbb{Q}_- \mapsto \mathbb{Q}_{+-} \oplus \mathbb{Q}_{-+}
\end{aligned}$$

The map $K_0(\mathbb{Q}[e]) \to K_0(\mathbb{Q}[\mathbb{Z}_2])$ is given by

$$\mathbb{Q} \mapsto \mathbb{Q}_+ \oplus \mathbb{Q}_-.$$

Counting, we find 68 inclusions between pairs of finite subgroups.

Rapidly finishing the calculation of $K_0(\mathbb{Q}\Gamma)$, we see that

$$8 \cdot K_0(\mathbb{Q}[(\mathbb{Z}_2)^2]) \oplus 12 \cdot K_0(\mathbb{Q}[\mathbb{Z}_2]) \oplus \cdot K_0(\mathbb{Q}[e]) \ = 8 \cdot \mathbb{Z}^4 \oplus 12 \cdot \mathbb{Z}^2 \oplus \mathbb{Z} = \mathbb{Z}^{57},$$

and that $K_0(\mathbb{Q}\Gamma)$ is the quotient of this by the 68 relators given above; the resulting 68×57 matrix is given to a computer algebra system (in this case, Sage [sag]), converted into Smith normal form, yielding the abelian invariants of the cokernel, and proving

$$K_0(\mathbb{Q}\Gamma) = \mathbb{Z}^{24} \oplus \mathbb{Z}_2.$$

□

This method of proof—relying on a computer to compute the cokernel—is not very illuminating, but the technique works in general: assuming the Farrell–Jones conjecture holds for Γ, there is an *algorithm* for computing $K_0(\mathbb{Q}\Gamma)$ provided we know the K-groups for the finite subgroups of Γ and the maps between them.

In this special case, a bit of work will shed more light on $K_0(\mathbb{Q}\Gamma)$. We can represent a class in $K_0(\mathbb{Q}\Gamma)$ as an eight-tuple of classes in $K_0(\mathbb{Q}[(\mathbb{Z}_2)^2])$, up to the equivalence relation given by the colimit. A representative for the element of order two in $K_0(\mathbb{Q}\Gamma)$ is

$$\begin{tikzcd}
& -\mathbb{Q}_{++} \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash] & & \mathbb{Q}_{-+} \arrow[dl, dash] \arrow[dd, dash] \\
\mathbb{Q}_{++} \arrow[rr, dash] \arrow[dd, dash] & & -\mathbb{Q}_{-+} \arrow[dd, dash] & \\
& \mathbb{Q}_{+-} \arrow[rr, dash] \arrow[dl, dash] & & -\mathbb{Q}_{--} \arrow[dl, dash] \\
-\mathbb{Q}_{+-} \arrow[rr, dash] & & \mathbb{Q}_{--} &
\end{tikzcd} \neq 0.$$

This element of $K_0(\mathbb{Q}\Gamma)$ does not vanish because the equivalence relation only modifies the module associated to a particular vertex by adding a module having $\dim_{\mathbb{Q}} = \pm 2$. We verify that this element actually is order two in Figure 4.2.

Question 4.5.14. Is there a group Γ for which the finiteness obstruction is torsion in $\tilde{K}_0(\mathbb{Q}\Gamma)$?

In fact, the group appearing in Lemma 4.5.13 is such an example, as shown in [KM91].

Figure 4.2: Two-torsion in $K_0(\mathbb{Q}\Gamma)$.

$$
\begin{tikzcd}
 & -2\cdot\mathbb{Q}_{++} \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash] & & +2\cdot\mathbb{Q}_{-+} \arrow[dl, dash] \arrow[dd, dash] & \\
+2\cdot\mathbb{Q}_{++} \arrow[rr, dash] \arrow[dd, dash] & & -2\cdot\mathbb{Q}_{-+} \arrow[dd, dash] & & = \\
 & +2\cdot\mathbb{Q}_{+-} \arrow[rr, dash] \arrow[dl, dash] & & -2\cdot\mathbb{Q}_{--} \arrow[dl, dash] & \\
-2\cdot\mathbb{Q}_{+-} \arrow[rr, dash] & & +2\cdot\mathbb{Q}_{--} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & +\mathbb{Q}_{+-}\oplus +\mathbb{Q}_{-+} \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash] & & -\mathbb{Q}_{++}\oplus -\mathbb{Q}_{--} \arrow[dl, dash] \arrow[dd, dash] & \\
-\mathbb{Q}_{+-}\oplus -\mathbb{Q}_{-+} \arrow[rr, dash] \arrow[dd, dash] & & +\mathbb{Q}_{++}\oplus +\mathbb{Q}_{--} \arrow[dd, dash] & & = \\
 & -\mathbb{Q}_{++}\oplus -\mathbb{Q}_{--} \arrow[rr, dash] \arrow[dl, dash] & & +\mathbb{Q}_{+-}\oplus +\mathbb{Q}_{-+} \arrow[dl, dash] & \\
+\mathbb{Q}_{++}\oplus +\mathbb{Q}_{--} \arrow[rr, dash] & & -\mathbb{Q}_{+-}\oplus -\mathbb{Q}_{-+} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash] & & 0 \arrow[dl, dash] \arrow[dd, dash] & \\
0 \arrow[rr, dash] \arrow[dd, dash] & & 0 \arrow[dd, dash] & & = 0. \\
 & 0 \arrow[rr, dash] \arrow[dl, dash] & & 0 \arrow[dl, dash] & \\
0 \arrow[rr, dash] & & 0 & &
\end{tikzcd}
$$

CHAPTER 5
SURGERY

5.1 Introduction

Uniform lattices, even when they contain torsion, satisfy PD($\mathbb{Q}$); a natural question then arises: to what extent is this rational Poincaré duality "explainable" as having arisen from some geometry? Does Γ satisfy because Γ is the fundamental group of a rational homology manifold having $\mathbb{Q}$-acyclic universal cover? In short, are such Γ also Mfld($\mathbb{Q}$) groups?

The following theorem asserts that, at least for uniform lattices with odd torsion, this is not the case—indeed, PD($\mathbb{Q}$) $\not\subset$ Mfld($\mathbb{Q}$).

Theorem 5.1.1. *Suppose Γ is a uniform lattice in a semisimple Lie group, and that Γ contains p-torsion for $p \neq 2$. Then there does not exist a $\mathbb{Q}$-homology manifold X with*

- *fundamental group $\pi_1 X = \Gamma$, and*
- *universal cover $\tilde{X}$ $\mathbb{Q}$-acyclic.*

One might hope that the proof would be analogous to the obstruction that the Euler characteristic provided with the Lefschetz fixed point theorem (Proposition 4.2.1), perhaps through an application of the G-signature formula. The trouble with this potential technique is that we do not have a G-signature formula for $\mathbb{Q}$-homology manifolds.

Instead, Theorem 5.1.1 will be proved via a calculation of controlled symmetric signatures (which basically amounts to proving a piece of the G-signature formula).

Sketch of Proof of Theorem 5.1.1. If there exists such an X as described by the theorem, then $\tilde{X}$ is a $\mathbb{Q}$-homology manifold, and therefore, $\tilde{X} \to \tilde{X}$ is a controlled $\mathbb{Q}$-Poincaré duality complex. This gives a class

$$\sigma^{\Gamma}_{\mathcal{Triv}}(\tilde{X}) \in \mathcal{H}^{\Gamma}_{\star}(E_{\mathcal{Triv}}\Gamma; \mathbb{L}) \otimes \mathbb{Q},$$

called the controlled symmetric signature.

The constant map $E\Gamma = E_{\mathcal{Triv}}\Gamma \to \bullet = E_{\mathcal{All}}\Gamma$ induces the assembly map

$$\mathrm{Asm} : \mathcal{H}^{\Gamma}_{\star}(E_{\mathcal{Triv}}\Gamma; \mathbb{L}) \to \mathcal{H}^{\Gamma}_{\star}(\bullet; \mathbb{L}) = L^{\star}(\mathbb{Q}\Gamma).$$

But the map $E\Gamma \to \bullet$ can be factored as $E\Gamma \to E_{\mathcal{Fin}}\Gamma \to \bullet$; these two maps induce

$$\begin{aligned}
\mathrm{Asm}^{\mathcal{Fin}}_{\mathcal{Triv}} &: \mathcal{H}^{\Gamma}_{\star}(E_{\mathcal{Triv}}\Gamma; \mathbb{L}) \to \mathcal{H}^{\Gamma}_{\star}(E_{\mathcal{Fin}}\Gamma; \mathbb{L}), \\
\mathrm{Asm}^{\mathcal{All}}_{\mathcal{Fin}} &: \mathcal{H}^{\Gamma}_{\star}(E_{\mathcal{Triv}}\Gamma; \mathbb{L}) \to \mathcal{H}^{\Gamma}_{\star}(\bullet; \mathbb{L}) = L^{\star}(\mathbb{Q}\Gamma),
\end{aligned}$$

so-called *partial assembly maps* factoring $\mathrm{Asm} = \mathrm{Asm}^{\mathcal{All}}_{\mathcal{Fin}} \circ \mathrm{Asm}^{\mathcal{Fin}}_{\mathcal{Triv}}$.

Assembling the controlled symmetric signature gives

$$\mathrm{Asm}\, \sigma^{\Gamma}_{\mathcal{Triv}}(\tilde{X}) = \sigma^{\star}(X) \in L^{\star}(\mathbb{Q}\Gamma),$$

which is the usual symmetric signature. But since Γ is a uniform lattice, we have, by Selberg's lemma, a torsion-free finite index subgroup $\pi < \Gamma$, with $B\pi$ a closed manifold. Thus, $E\pi$ is a manifold and model for $E_{\mathcal{Fin}}\Gamma$, giving

$$\sigma^{\Gamma}_{\mathcal{Fin}}(E\pi) \in \mathcal{H}^{\Gamma}_{\star}(E_{\mathcal{Fin}}\Gamma; \mathbb{L})$$

with $\mathrm{Asm}^{\mathcal{All}}_{\mathcal{Fin}}\, \sigma^{\Gamma}_{\mathcal{Fin}}(B\pi) = \sigma(X) \in L^{\star}(\mathbb{Q}\Gamma)$. The Novikov conjecture holds for Γ, meaning $\mathrm{Asm}^{\mathcal{All}}_{\mathcal{Fin}} \otimes \mathbb{Q}$ is injective, hence it must be that

$$\mathrm{Asm}^{\mathcal{Fin}}_{\mathcal{Triv}}\, \sigma^{\Gamma}_{\mathcal{Triv}}(X) = \sigma^{\Gamma}_{\mathcal{Fin}}(B\pi) \in \mathcal{H}^{\Gamma}_{\star}(E_{\mathcal{Fin}}\Gamma; \mathbb{L}) \otimes \mathbb{Q}.$$

By restricting to a small neighborhood of a fixed point, we will find that this is impossible (i.e., the symmetric signature of a free action cannot give rise to the symmetric signature for a fixed point). □

We need the fact that Γ is a uniform lattice, in order to apply the Novikov conjecture; but this method can be considered whenever Γ is a group extension

$$1 \to \pi \to \Gamma \to G \to 1$$

with $B\pi$ having the homotopy type of a closed manifold, and G finite. S. Weinberger has analyzed the situation when $\Gamma = \pi \times G$ (see [Wei86a], [Wei86b], [Wei85]).

5.2 Background

The classic reference for surgery is [Wal99]; the stratified situation is worked out by Weinberger in [Wei94], and the algebraic machinery is developed by Ranicki in [Ran92].

5.2.1 Families of subgroups

For a group G, the universal G-space EG is a well-known object in homotopy theory; any free G-CW complex admits a G-map to EG. It will be useful to have terminal objects for G-spaces with not necessarily free actions, but actions having some restriction on their isotropy.

Definition 5.2.1. A *family of subgroups* $\mathcal{F}$ of a group G is a collection of subgroups of G, closed under conjugation and finite intersection.

Of particular importance are

$$\begin{aligned} \mathcal{All} &= \{ \text{ all subgroups } \}, \\ \mathcal{Fin} &= \{ \text{ finite subgroups } \}, \text{ and} \\ \mathcal{Triv} &= \{ \text{ the trivial subgroup } \}. \end{aligned}$$

But others, like $\mathcal{Cyc}$ and $\mathcal{VCyc}$, the cyclic and the virtual cyclic subgroups, respectively, are also important.

Definition 5.2.2. Let $\mathcal{F}$ be a family of subgroups of G; an $\mathcal{F}$-G-CW-complex is a G-CW-complex with isotropy groups in $\mathcal{F}$. An *$\mathcal{F}$-classifying space* for G is an $\mathcal{F}$-G-CW-complex $E_{\mathcal{F}}G$ so that the fixed set $(E_{\mathcal{F}}G)^H$ is weakly contractible for any $H \in \mathcal{F}$. Equivalently, $E_{\mathcal{F}}G$ is a terminal object in the category of $\mathcal{F}$-G-CW-complexes.

It is possible to construct $E_{\mathcal{F}}G$ for any family of subgroups $\mathcal{F}$ of G. For $\mathcal{All}$ and $\mathcal{Triv}$, these are familiar examples.

Example 5.2.3. $E_{\mathcal{All}}G = G/G = \bullet$, and $E_{\mathcal{Triv}}G = EG$. The classifying space $E_{\mathcal{Fin}}G$ classifies *proper* actions. Some authors denote $E_{\mathcal{Fin}}G$ by $\underline{E}G$.

5.2.2 Equivariant homology theory

Analogous to the Eilenberg–Steenrod axioms for homology [ES45], there are axioms characterizing an equivariant homology theory. We will be following the presentation given in [DL98].

In particular, an equivariant homology theory functorially assigns, to a G-CW-pair (X, A), the R-modules $\mathcal{H}_\star^G(X, A)$. We write $\mathcal{H}_\star^G(X)$ to mean $\mathcal{H}_\star^G(X, \varnothing)$.

There is a natural transformation

$$\partial : \mathcal{H}_\star^G(X, A) \to \mathcal{H}_{\star-1}^G(A),$$

and the sequence of functors $\mathcal{H}_\star^G$ satisfies *equivariant Eilenberg–Steenrod axioms.*

Homotopy invariance. If

$$f_0, f_1 : (X, A) \to (Y, B)$$

are G-homotopic maps of G-CW-pairs, then $\mathcal{H}_\star^G(f_0) = \mathcal{H}_\star^G(f_1)$.

Exactness. Given a G-CW-pair (X, A), there is a long exact sequence

$$\cdots \to \mathcal{H}^G_{\star+1}(X, A) \xrightarrow{\partial} \mathcal{H}^G_\star(A) \xrightarrow{i_\star} \mathcal{H}^G_\star(X) \xrightarrow{j_\star} \mathcal{H}^G_\star(X, A) \xrightarrow{\partial} \cdots$$

where $i : A \to X$ and $j : X \to (X, A)$ are inclusions.

Excision. For a G-CW-pair (X, A) and a cellular G-map $f : A \to B$, the natural map

$$\mathrm{exc} : \mathcal{H}^G_\star(X, A) \xrightarrow{\cong} \mathcal{H}^G_\star(X \cup_f B, B)$$

is an isomorphism.

Additivity. For a family $\{X_i\}_{i\in I}$ of G-CW-complexes, the natural map

$$\bigoplus_{i\in I} \mathcal{H}^G_\star(X_i) \xrightarrow{\cong} \mathcal{H}^G_\star\left(\bigcup_{i\in I} X_i\right)$$

is an isomorphism.

As a consequence of the Eilenberg–Steenrod axioms, non-equivariant generalized homology theories have a suspension isomorphism;we analogously have an equivariant *suspension isomorphism*

$$\mathcal{H}^G_n(X) \cong \mathcal{H}^G_{n+1}(\Sigma X),$$

provided $n \geq 1$.

The equivariant homology theories for different groups are often related.

Definition 5.2.4. Suppose $\varphi : H \to G$ is a group homomorphism, and X is an H-space. Then the *induction* of X with φ is the G-space

$$\mathrm{ind}_\varphi X = (G \times X)/H$$

where H acts by $(g, x) \cdot h = (g\,\varphi(h), h^{-1}x)$ for $h \in H$ and $(g, x) \in G \times X$.

We use induction to relate equivariant G-homology theories for groups G and H.

Definition 5.2.5. An equivariant homology theory $\mathcal{H}_\star^?$ is an equivariant homology theory $\mathcal{H}_\star^G$ for each group G, related by an *induction structure*.

An induction structure is the following: given a group homomorphism $\varphi : H \to G$ and an H-CW-pair (X, A) on which $\ker \varphi$ acts freely, there is a natural isomorphism

$$\mathrm{ind}_\varphi : \mathcal{H}_\star^H(X, A) \xrightarrow{\cong} \mathcal{H}_n^G(\mathrm{ind}_\varphi(X, A)).$$

Remark 5.2.6. Just as homology theories are encoded by spectra [Ada74], equivariant homology theories correspond to equivariant spectra (in fact, spectra over an orbit category—the same machinery that appeared in Section 4.3).

From now on, we will denote by $\mathbb{L}_R$ the equivariant spectrum we are interested in, namely, $\mathbb{L}_R^{\langle -\infty \rangle}$. If we do not mention the ring, we mean $R = \mathbb{Q}$ (in spite of the usual convention that $R = \mathbb{Z}$).

5.2.3 Controlled L-theory

Controlled L-theory for algebraic Poincaré duality complexes with $\mathbb{Q}$-coefficients is developed in [RY06]. Our use of the controlled theory is quite soft, so we merely record the few facts that we will be using.

A chapter introducing controlled topology appears in [Wei94]; the basic idea is to do topology over a metric space, and measure the sizes of various operations (e.g., homotopies) in that metric space. As an example, an R-homology n-manifold (by definition) has the same local R-homology as $\mathbb{R}^n$, and so it has a "local" Poincaré duality at all scales.

Lemma 5.2.7. *Let X be a compact R-homology manifold. Then $\tilde{X}$ is an ϵ-controlled R-Poincaré duality complex over itself, for all $\epsilon > 0$.*

A consequence of Theorem 8.5 in [RY06] is the following.

Theorem 5.2.8. *Let X be a finite polyhedron and suppose $M \to X$ is a fibration with path-connected fiber F and* $\mathrm{Wh}(\pi_1 F \times \mathbb{Z}^k) = 0$ *for all $k \geq 0$. Then the controlled L-theory of $M \to X$ is a homology theory.*

As in [Ros06], what we require is the vanishing of lower algebraic K-theory. With the $\langle -\infty \rangle$ decoration, we avoid the K-theoretic difficulties, and controlled L-theory is a homology theory.

5.2.4 Symmetric signature

A basic invariant of a Poincaré duality complex is its *signature*. The symmetric signature is a more refined notion.

Definition 5.2.9. Suppose X is an R-Poincaré duality complex; by definition, this means that the chain complex $C_\star(\tilde{X})$ is an algebraic Poincaré complex (see [Ran01]). The *symmetric signature* is the class of $C_\star(X)$ in the cobordism group of (symmetric) algebraic Poincaré complexes $L^\star(\mathbb{Q}[\pi_1 X])$.

This can also be done in the controlled setting.

Definition 5.2.10. Suppose X is a controlled R-Poincaré duality complex over a metric space M. Then the *controlled symmetric signature* is the class

$$\sigma^\star_M(X) \in H_n(M; \mathbb{L}_R)$$

corresponding to X viewed as an M-controlled R-Poincaré duality complex.

For a particular application, consider this: an R-homology manifold, being a controlled Poincaré duality complex over itself, gives rise to a controlled symmetric signature.

5.2.5 Assembly Maps

The Γ-map $E\Gamma \to \bullet$ induces a map on $\mathbb{L}$-homology

$$H_\star(B\Gamma;\mathbb{L}_R) = H_\star^\Gamma(E\Gamma;\mathbb{L}_R) \xrightarrow{\mathrm{Asm}} H_\star^\Gamma(\bullet;\mathbb{L}_R) = L_\star(R\Gamma),$$

called the *assembly map*, given a nice geometric interpretation by Quinn [Qui95].

The Novikov conjecture (on homotopy invariance of higher signatures [Nov70]) is implied by the following conjecture.

Conjecture 5.2.11. The rational assembly map

$$\mathrm{Asm} \otimes \mathbb{Q} : H_\star(B\Gamma;\mathbb{L}_\mathbb{Z}) \otimes \mathbb{Q} \to L_\star(\mathbb{Z}\Gamma) \otimes \mathbb{Q}$$

is injective.

A nice overview on the Novikov conjecture appears in [CW06].

In many cases, the assembly map is also integrally injective, even if $\mathbb{Z}$ is replaced with $\mathbb{Q}$. In particular, the work appearing in [Ros06] implies that

$$\mathrm{Asm} : H_\star(B\Gamma;\mathbb{L}_\mathbb{Q}) \to L_\star(\mathbb{Q}\Gamma)$$

is injective for Γ a lattice (possibly with torsion) in a Lie group. This will be important in the sequel.

There are other characterizations of assembly maps; Weiss–Williams, in particular, characterize assembly maps as an excisive approximation to an arbitrary functor [WW95]; this theory has been generalized to the equivariant case in [DL98], and so, the forget control map is the usual assembly map Asm.

5.3 Proof of Theorem 5.1.1

Assume that there exists X, a $\mathbb{Q}$-homology manifold, with $\tilde{X}$ rationally acyclic and $\pi_1 = \Gamma$, for a lattice Γ in a semisimple Lie group, where Γ contains p-torsion for $p \neq 2$. We will show that this is impossible (proving Theorem 5.1.1).

It suffices to consider the case where Γ is an extension

$$1 \to \pi \to \Gamma \to \mathbb{Z}_p \to 1$$

with $p \neq 2$ and π a torsion-free lattice. For an arbitrary lattice Γ with p-torsion, by Selberg's lemma, Γ admits a map φ to a finite group H having $\ker \varphi$ torsion-free; since Γ has p-torsion, we also have $H \supset \mathbb{Z}_p$. Let $\Gamma' = \varphi^{-1}(\mathbb{Z}_p)$. Then Γ' is a finite index subgroup of Γ, so if Γ' is not the fundamental group of a closed $\mathbb{Q}$-homology manifold with $\mathbb{Q}$-acyclic universal cover, neither is Γ.

Since X is a $\mathbb{Q}$-homology manifold, it is an arbtirarily well-controlled $\mathbb{Q}$-Poincaré duality complex over itself; we get a controlled symmetric signature

$$\sigma^{\Gamma}_{\mathcal{T}riv}(X) \in \mathcal{H}^{\Gamma}_n(\tilde{X}; \mathbb{L}).$$

Using the Γ-map $\tilde{X} \to E\Gamma$, we regard $\sigma^{\Gamma}_{\mathcal{T}riv}(X)$ as an element of $\mathcal{H}^{\Gamma}_n(E\Gamma; \mathbb{L})$. In fact, the induced map

$$\mathcal{H}^{\Gamma}_n(\tilde{X}; \mathbb{L}) \to= \mathcal{H}^{\Gamma}_n(E\Gamma; \mathbb{L})$$

is a rational isomorphism (by an equivariant Atiyah-Hirzebruch spectral sequence), but we will not need this fact.

The class $\sigma^{\Gamma}_{\mathcal{T}riv}(\tilde{X})$ assembles to give $\sigma^{\star}(X) \in L^{\star}(\mathbb{Q}\Gamma)$; in fact, we have another object which assembles to the same (uncontrolled) symmetric signature.

Lemma 5.3.1. $\mathrm{Asm}\, \sigma^{\Gamma}_{\mathcal{T}riv}(\tilde{X}) = \mathrm{Asm}^{\mathcal{A}ll}_{\mathcal{F}in}\, \sigma^{\Gamma}_{\mathcal{F}in}(E_{\mathcal{F}in}\Gamma)$

Since the integral Novikov conjecture holds for Γ (see Section 5.2.5), the assembly map is injective, and therefore, the partial assembly map $\mathrm{Asm}^{\mathcal{A}ll}_{\mathcal{F}in}$ is also injective. As a result,

Corollary 5.3.2. $\mathrm{Asm}^{\mathcal{F}in}_{\mathcal{T}riv}\, \sigma^{\Gamma}_{\mathcal{T}riv}(\tilde{X}) = \sigma^{\Gamma}_{\mathcal{F}in}(E_{\mathcal{F}in}\Gamma)$.

In other words, the existence of the rational homology manifold X with $\pi_1 X = \Gamma$ has resulted in a particular controlled symmetric signature lifting from $E_{\mathcal{F}in}\Gamma$ to $E\Gamma$.

There is actually one issue here: rationally, it is possible to modify the fundamental class without losing Poincaré duality.

Definition 5.3.3 ([Wei86b]). Define an endomorphism of symmetric algebraic Poincaré complexes,

$$F_n : L^\star(\mathbb{Q}\Gamma) \to L^\star(\mathbb{Q}\Gamma)$$

by multiplying the fundamental class of an algebraic Poincaré complex by n.

The equalities we assert on symmetric signatures are only true after applying some F_n; but F_n is an isomorphism (with inverse F_n), so this will not be an issue.

5.3.1 Induction

Recall that there is a map $\varphi : \Gamma \to \mathbb{Z}_p$ with $\ker\varphi = \pi$. Note that

$$\begin{aligned} \operatorname{ind}_\varphi E\Gamma &= B\pi \times E\mathbb{Z}_p, \\ \operatorname{ind}_\varphi E_{\mathcal{F}in}\Gamma &= B\pi, \end{aligned}$$

and since $\ker\varphi$ acts freely on both $E\Gamma$ and $E_{\mathcal{F}in}\Gamma$, we get isomorphisms as described in Definition 5.2.5,

$$\begin{aligned} \mathcal{H}^\Gamma_\star(E\Gamma;\mathbb{L}) &\xrightarrow{\cong} \mathcal{H}^{\mathbb{Z}_p}_n(B\pi \times E\mathbb{Z}_p;\mathbb{L}), \text{ and} \\ \mathcal{H}^\Gamma_n(E_{\mathcal{F}in}\Gamma;\mathbb{L}) &\xrightarrow{\cong} \mathcal{H}^{\mathbb{Z}_p}_n(B\pi;\mathbb{L}). \end{aligned}$$

We can put together all the pieces we have thus far in a diagram.

$$\begin{tikzcd} \mathcal{H}^\Gamma_n(\tilde{X};\mathbb{L}) \arrow[r] & \mathcal{H}^\Gamma_n(E\Gamma;\mathbb{L}) \arrow[r, "\cong"] \arrow[d] & \mathcal{H}^{\mathbb{Z}_p}_n(B\pi \times E\mathbb{Z}_p;\mathbb{L}) \arrow[d] \\ & \mathcal{H}^\Gamma_n(E_{\mathcal{F}in}\Gamma;\mathbb{L}) \arrow[r, "\cong"] \arrow[d] & \mathcal{H}^{\mathbb{Z}_p}_n(B\pi;\mathbb{L}) \\ & \mathcal{H}^\Gamma_n(\bullet;\mathbb{L}) & \end{tikzcd}$$

So we have $\sigma^{\Gamma}_{\mathcal{F}in}(B\pi) \in \mathcal{H}^{\mathbb{Z}_p}_n(B\pi;\mathbb{L})$, and this lifts to $\sigma^{\Gamma}_{\mathcal{T}riv}(X) \in \mathcal{H}^{\mathbb{Z}_p}_n(B\pi \times E\mathbb{Z}_p;\mathbb{L})$. We prove that this is impossible.

No finite group can act freely on an finite aspherical complex, so the fixed set $(B\pi)^{\mathbb{Z}_p}$ is nonempty; let U be a $\mathbb{Z}_p$-equivariant regular neighborhood of some fixed point $x \in B\pi$. Then the inclusion map $j : B\pi \to (B\pi, B\pi - U)$ and excision gives

$$\begin{tikzcd}
\sigma^{\Gamma}_{\mathcal{T}riv}(X) \arrow[r, mapsto] & \sigma^{\Gamma}_{\mathcal{F}in}(B\pi) \\[-15pt]
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\[-15pt]
\mathcal{H}^{\mathbb{Z}_p}_n(B\pi \times E\mathbb{Z}_p;\mathbb{L}) \arrow[r] \arrow[d, "j_\star"] & \mathcal{H}^{\mathbb{Z}_p}_n(B\pi;\mathbb{L}) \arrow[d, "j_\star"] \\
\mathcal{H}^{\mathbb{Z}_p}_n(B\pi \times E\mathbb{Z}_p, (B\pi - U) \times E\mathbb{Z}_p;\mathbb{L}) \arrow[r] \arrow[d, "\mathrm{exc}"] & \mathcal{H}^{\mathbb{Z}_p}_n(B\pi, B\pi - U;\mathbb{L}) \arrow[d, "\mathrm{exc}"] \\
\mathcal{H}^{\mathbb{Z}_p}_n(U \times E\mathbb{Z}_p, \partial U \times E\mathbb{Z}_p;\mathbb{L}) \arrow[r] & \mathcal{H}^{\mathbb{Z}_p}_n(U, \partial U;\mathbb{L})
\end{tikzcd}$$

The fixed point $x \in U$ is not necessarily isolated; however, $U = \Sigma^k V$ for some $\mathbb{Z}_p$-space V with an isolated fixed point. This allows us to apply some suspension isomorphisms, that is,

$$\begin{tikzcd}
\mathrm{exc}\, j_\star \sigma^{\Gamma}_{\mathcal{T}riv}(X) \arrow[r, mapsto] & \mathrm{exc}\, j_\star \sigma^{\Gamma}_{\mathcal{F}in}(B\pi) \\[-15pt]
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\[-15pt]
\mathcal{H}^{\mathbb{Z}_p}_n(U \times E\mathbb{Z}_p, \partial U \times E\mathbb{Z}_p;\mathbb{L}) \arrow[r] \arrow[d, equal] & \mathcal{H}^{\mathbb{Z}_p}_n(U, \partial U;\mathbb{L}) \arrow[d, equal] \\
\mathcal{H}^{\mathbb{Z}_p}_n(\Sigma^k V \times E\mathbb{Z}_p, \partial \Sigma^k V \times E\mathbb{Z}_p;\mathbb{L}) \arrow[r] \arrow[d] & \mathcal{H}^{\mathbb{Z}_p}_n(\Sigma^k V, \partial \Sigma^k V;\mathbb{L}) \arrow[d, "\cong"] \\
\mathcal{H}^{\mathbb{Z}_p}_{n-k}(V \times E\mathbb{Z}_p, \partial V \times E\mathbb{Z}_p;\mathbb{L}) \arrow[r] & \mathcal{H}^{\mathbb{Z}_p}_{n-k}(V, \partial V;\mathbb{L})
\end{tikzcd}$$

The issue, then, is determining the extent to which elements of $\mathcal{H}^{\mathbb{Z}_p}_{n-k}(V, \partial V;\mathbb{L})$ lift to $\mathcal{H}^{\mathbb{Z}_p}_{n-k}(V \times E\mathbb{Z}_p, \partial V \times E\mathbb{Z}_p;\mathbb{L})$. To see that this is impossible, we reinterpret these homology groups. First, the left hand side becomes

$$
\begin{aligned}
\mathcal{H}^{\mathbb{Z}_p}_{n-k}(V, \partial V; \mathbb{L}) &= \operatorname{coker}\left(\mathcal{H}^{\mathbb{Z}_p}_{n-k}(\partial V; \mathbb{L}) \to \mathcal{H}^{\mathbb{Z}_p}_{n-k}(V; \mathbb{L})\right) \\
&= \operatorname{coker}\left(\mathcal{H}_{n-k}(\partial V/\mathbb{Z}_p; \mathbb{L}) \to \mathcal{H}^{\mathbb{Z}_p}_{n-k}(\bullet; \mathbb{L})\right) \\
&= \operatorname{coker}\left(\mathcal{H}_{n-k}(\partial V/\mathbb{Z}_p; \mathbb{L}) \to L_{n-k}(\mathbb{Q}[\mathbb{Z}_p])\right)
\end{aligned}
$$

The right hand side can be interpreted as

$$
\begin{gathered}
\mathcal{H}^{\mathbb{Z}_p}_{n-k}(V \times E\mathbb{Z}_p, \partial V \times E\mathbb{Z}_p; \mathbb{L}) = \\
\operatorname{coker}\left(\mathcal{H}^{\mathbb{Z}_p}_{n-k}(\partial V \times E\mathbb{Z}_p; \mathbb{L}) \to \mathcal{H}^{\mathbb{Z}_p}_{n-k}(V \times E\mathbb{Z}_p; \mathbb{L})\right) = \\
\operatorname{coker}\left(\mathcal{H}_{n-k}(\partial V/\mathbb{Z}_p; \mathbb{L}) \to \mathcal{H}_{n-k}(B\mathbb{Z}_p; \mathbb{L})\right).
\end{gathered}
$$

In both these cases, we made use of the $\mathbb{Z}_p$-homotopy invariance, exactness for the pair, and the isomorphisms coming from the induction structure.

Following all the maps, the fact that $\sigma^{\Gamma}_{\mathcal{Fin}}(B\pi)$ lifts to $\sigma^{\Gamma}_{\mathcal{Triv}}(X)$ yields

$$
\begin{tikzcd}
 & {j_\star}^{-1}\partial_\star\Sigma^{-k} \operatorname{exc} j_\star \sigma^{\Gamma}_{\mathcal{Triv}}(X) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \arrow[dddd, mapsto, bend left=60] \\
\mathcal{H}_{n-k}(\partial V/\mathbb{Z}_p; \mathbb{L}) \arrow[r] \arrow[d, equal] & \mathcal{H}_{n-k}(B\mathbb{Z}_p; \mathbb{L}) \arrow[d] \\
\mathcal{H}_{n-k}(\partial V/\mathbb{Z}_p; \mathbb{L}) \arrow[r] & L_{n-k}(\mathbb{Q}[\mathbb{Z}_p]) \arrow[d, phantom, "\rotatebox{90}{$\ni$}"] \\
 & \\
 & {j_\star}^{-1}\partial_\star\Sigma^{-k} \operatorname{exc} j_\star \sigma^{\Gamma}_{\mathcal{Fin}}(B\pi)
\end{tikzcd}
$$

To analyze V, we build a manifold M with a $\mathbb{Z}_p$ action, having isolated fixed points m_i for $i \in I$, all of which have the same normal representation V; this can be done by taking a product of surfaces. Let V_i be an equivariant regular neighborhood of the fixed point m_i. Then $\mathbb{Z}_p$ acts freely on $M - \bigsqcup_i V_i$, inducing the rightmost vertical map in the following diagram.

$$\begin{tikzcd}
\mathcal{H}_\star^{\mathbb{Z}_p}(M - \bigsqcup_i V_i; \mathbb{L}) \arrow[r] \arrow[d] & \mathcal{H}_\star^{\mathbb{Z}_p}(M; \mathbb{L}) \arrow[r] \arrow[d] & \mathcal{H}_\star^{\mathbb{Z}_p}(M, M - \bigsqcup_i V_i; \mathbb{L}) \arrow[d, "\cong"] \\
\mathcal{H}_\star(B\mathbb{Z}_p; \mathbb{L}) \arrow[r] & \mathcal{H}_\star^{\mathbb{Z}_p}(\bullet; \mathbb{L}) \arrow[d, "\cong"] & \bigoplus_i \mathcal{H}_\star(V_i, \partial V_i; \mathbb{L}) \\
& L_\star(\mathbb{Q}[\mathbb{Z}_p]) &
\end{tikzcd}$$

The controlled symmetric signature $\sigma_\star(M \downarrow M) \in \mathcal{H}_\star^{\mathbb{Z}_p}(M; \mathbb{L})$ maps to

$$\bigoplus_i \mathcal{H}_\star(V_i, \partial V_i; \mathbb{L}),$$

as the sum of ρ invariants of V_i. If the controlled symmetric signature vanishes under this map, then it comes from $\mathcal{H}_\star^{\mathbb{Z}_p}(M - \bigsqcup_i V_i; \mathbb{L})$, which gives the lift on the bottom row of the diagram and therefore a vanishing of the G-signature in

$$\mathrm{RO}(\mathbb{Z}_p)/\langle\text{regular representation}\rangle.$$

But by Theorem 14E.7 in [Wal99], the ρ invariant is not a multiple of the regular representation. This concludes the proof of Theorem 5.1.1

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